3 Which is faster, going up or coming down?

Suppose you throw a ball into the air. If one neglects the air resistance, then it takes the same time for the ball to reach the highest point of its trajectory and to fall down from that point to the Earth. Now, if one does take the air resistance into account, which way will it take the ball longer to go, up or down?

It can be shown, both rigorously and at the intuitive level, that it will take longer for the ball to fall than to rise. This holds for any law of air resistance, as long as the motion of the ball is strictly vertical (i.e., one-dimensional) and the air resistance depends only on the ball’s velocity. For details see the article by F. Brauer posted online.

In this lecture, we will answer the above question in a much restricted setting, namely: when the air resistance is very small (compared to the gravity) and, moreover, when its magnitude is a simple linear or quadratic function of the velocity. The reason we will study this restricted setting is that we will illustrate some basic steps of the perturbative approach to solving algebraic and differential equations. We will also see some applications of the Taylor series expansion of functions.

3.1 The exact model

Let a ball with mass $m$ be moving vertically (up or down), and its initial velocity (pointing up) be $v_0$. We assume that the only two forces acting on the ball are the gravity and the air resistance, with the latter being directed oppositely to the direction of the ball’s motion.

![Diagram of forces](image)

Projecting Newton’s Second Law

$$m\ddot{a} = \sum \vec{F}$$

on the $y$-axis, we have the following equations (see the figure above).

**Going up, $v > 0$:**

$$m \frac{dv}{dt} = -mg - F_{\text{air}};$$

**Going down: $v < 0$:**

$$m \frac{dv}{dt} = -mg + F_{\text{air}}.$$

(3.1a)  

(3.1b)
We begin by taking $F_{\text{air}}$ to be proportional to the first power of the velocity, i.e.

$$F_{\text{air}} = D|v|.$$  \hfill (3.2)

It is known that for small bodies (e.g., a tennis ball) with not very large velocities (e.g., falling from the roof of a few-story building), (3.2) is a good model. On the other hand, for larger objects having larger velocities (e.g., a skydiver or a parachutist), the air resistance is proportional to the square of the velocity:

$$|F_{\text{air}}| = Dv^2.$$  \hfill (3.3)

See Sections 2 and 3 of the article by L. Long and H. Weiss posted online, if you want to see some physical arguments behind this.

For now, we focus on model (3.2). In more detail, it can be written as:

$$v > 0 \quad F_{\text{air}} = D \cdot v (> 0)$$
$$v < 0 \quad F_{\text{air}} = D \cdot (-v) (> 0).$$

Substitution of this into (3.1) yields:

$$m \frac{dv}{dt} = -mg - Dv$$  \hfill (3.4)

for both the upward and downward motions.

It is customary to nondimensionalize equations. For (3.4), we do this in two steps. First, dividing by $m$ yields

$$\frac{dv}{dt} = -g - \frac{D}{m}v.$$  

Second, let us introduce a new variable $\tau = gt$. Then, using the Chain Rule, we have:

$$\frac{dv}{dt} = \frac{dv}{d\tau} \cdot \frac{d\tau}{dt} = \frac{dv}{d\tau} \cdot g.$$  

Substituting this into the previous equation and cancelling by $g$, we obtain:

$$\frac{dv}{d\tau} = -1 - \frac{D}{mg}v.$$  

Finally, we use the notation introduced in Lecture 2:

$$\frac{dv}{d\tau} \equiv \dot{v}.$$  

Then the nondimensional form of Eq. (3.4) is:

$$\dot{v} = -1 - Kv,$$  \hfill (3.5)

where $K \equiv D/(mg)$. Note that due to the change of variables from $t$ to $\tau$, the usual equation $\frac{dy}{dt} = v$ now has the form

$$\dot{y} = \frac{1}{g}v.$$  \hfill (3.6)

The initial conditions for (3.5) and (3.6) are:

$$v(0) = v_0, \quad y(0) = y_0.$$  \hfill (3.7)
(In the specific case we are considering, \( y_0 = 0 \).) When the air resistance is absent (\( K = 0 \) in (3.5)), we get the usual formulae from (3.5), (3.6), and (3.7):

\[
\begin{align*}
v &= v_0 - \tau \\
y &= y_0 + \frac{1}{g} \left( v_0 \tau - \frac{\tau^2}{2} \right).
\end{align*}
\]

(Verify that they are equivalent to the form you are used to.)

Now let us consider the case \( K \neq 0 \) (\( K > 0 \)). Equation (3.5) can be solved by separation of variables:

\[
\frac{dv}{d\tau} = -1 - Kv
\]

\[
\quad \Rightarrow \quad \int \frac{dv}{1+Kv} = -\tau + C \quad (C = \text{const})
\]

\[
\quad \Rightarrow \quad \frac{1}{K} \ln(1 + Kv) = -\tau + C
\]

(verify). Using the first initial condition for \( v \) in (3.7), we find \( C \):

\[
C = \frac{1}{K} \ln(1 + Kv_0).
\]

Finally, we solve for \( v \) to obtain (verify):

\[
v = \frac{1}{K} \left( (1 + Kv_0)e^{-K\tau} - 1 \right).
\]

(3.9)

Using now (3.6) and the second initial condition in (3.7), we obtain (verify):

\[
y = y_0 + \frac{1}{gK} \left( (1 + Kv_0)\frac{1 - e^{-K\tau}}{K} - \tau \right).
\]

(3.10)

**Question:** We have obtained an answer. How do we know that we haven’t made a mistake?

**Answer:** Use sanity check — verify if the limiting cases make sense.

There are two limiting cases: \( K \gg 1 \) (very large) and \( K \ll 1 \) (very small). In the former case, we expect that the ball will go up by a small amount only (why?), and in the latter case we expect that the answer is close to that given by (3.8).

\( K \gg 1 \)

For any fixed \( \tau \) (i.e. when \( \tau \) is in no way related to \( K \)), (3.10) yields:

\[
y_{|K\gg1} = y_0 + \frac{1}{gK} \left( \frac{1 + Kv_0}{K} \frac{1 - e^{-K\tau}}{K} - \tau \right) \approx y_0 + \frac{1}{gK} (v_0 - \tau),
\]

i.e., indeed, the elevation of the ball is very small.

Before we consider the other limiting case, let us introduce a new notation. Suppose \( \varepsilon \) is a small number: \( \varepsilon \ll 1 \). Then one says that a function \( f(\varepsilon) \) is \( O(\varepsilon) \) if

\[
\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\varepsilon} = \text{const} \neq 0.
\]
For example:
\[ \varepsilon + 100\varepsilon^2 = O(\varepsilon), \quad \sin \varepsilon = O(\varepsilon), \quad \frac{15\varepsilon}{2 + 4\varepsilon} = O(\varepsilon). \]

Similarly, one defines \( O(\varepsilon^2), O(\varepsilon^3), \) etc:
\[ \lim_{\varepsilon \to 0} \frac{O(\varepsilon^n)}{\varepsilon^n} = \text{a nonzero number}. \]

The following “arithmethic” rules apply to the \( O(\cdot) \) notation:
\[
\begin{align*}
O(\varepsilon) + O(\varepsilon) &= O(\varepsilon) \\
O(\varepsilon) - O(\varepsilon) &= O(\varepsilon) \\
\text{const} \cdot O(\varepsilon) &= O(\varepsilon) \\
O(\varepsilon) \pm O(\varepsilon^2) &= O(\varepsilon) \\
\varepsilon \cdot O(\varepsilon) &= O(\varepsilon^2) \\
\frac{1}{\varepsilon} \cdot O(\varepsilon) &= O(1).
\end{align*}
\]

The generalization of these rules to other \( O(\varepsilon^n) \) is obvious.

Thus, returning to our analysis, we can say that when \( K \gg 1 \),
\[ y - y_0 = O\left(\frac{1}{K}\right). \]

Now let us consider the other limiting case.
\[ K \ll 1 \]
Recall that we want to confirm that in the limit \( K \to 0 \), Eq. (3.10) reduces to Eq. (3.8b). To this end, use the Maclaurin series for \( e^x \),
\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \]
to expand the expression in the large parentheses in (3.10) up to \( O(K^2) \) (you will see why in a moment):
\[
(1 + Kv_0) \frac{1 - (1 - K\tau + \frac{K^2\tau^2}{2} + O(K^3))}{K} - \tau = \\
(1 + Kv_0) \cdot \left(\tau - \frac{K\tau^2}{2} + O(K^2)\right) - \tau = \\
\tau - \frac{K\tau^2}{2} + O(K^2) + Kv_0\tau + O(K^2) + O(K^3) - \tau = \\
K\left(v_0\tau - \frac{\tau^2}{2}\right) + O(K^2).
\]

Substituting this back into (3.10), we find:
\[
y = y_0 + \frac{1}{gK} \left(K(v_0\tau - \frac{\tau^2}{2}) + O(K^2)\right) \\
= y_0 + \frac{1}{g} \left(v_0\tau - \frac{\tau^2}{2}\right) + O(K).
\]
Thus, indeed, the $O(1)$-term in the above expression coincides with (3.8b). This indicates that our answer (3.10) is, most likely, correct.

To conclude this section, let us answer the question posed in the title of this lecture, for model (3.5) (and hence for its solution (3.9), (3.10)).

It is easy to find the “time”, $\tau_m$, needed for the ball to reach the maximum elevation: just set $v = 0$ in (3.9). Then (verify):

$$\tau_m = \frac{1}{K} \ln(1 + Kv_0). \tag{3.11}$$

However, it is not possible to find analytically the time $\tau_1$ when the ball hits the ground, because it is not possible to solve analytically the transcendental equation (3.10) whose l.h.s. is set to 0.

Nonetheless, we can still answer our question by evaluating $y(2\tau_m)$. Indeed, if $y(2\tau_m) > 0$, then the ball is still in the air when $\tau = \tau_m + \tau_m$, i.e. going down is slower than going up. If, on the other hand, $y(2\tau_m) < 0$, then the ball has already hit the ground before $\tau = \tau_m + \tau_m$, so that in this case we would conclude that going down is faster.

So, we compute (setting $y_0 = 0$):

$$y(2\tau_m) = \frac{1}{gK} \left( (1 + Kv_0) \frac{1 - (e^{-K\tau_m})^{2}}{K} - 2\tau_m \right)$$

use (3.11)

$$= \frac{1}{gK} \left( \frac{1 + Kv_0}{K} \right) \left( 1 - \frac{1}{(1 + Kv_0)^2} \right) - \frac{2}{K} \ln(1 + Kv_0).$$

Let us denote $1 + Kv_0 \equiv x (> 1)$. Then $y(2\tau_m) = \frac{1}{gK} \left( x - \frac{1}{x} - 2 \ln x \right)$ (verify).

The r.h.s. of this expression is a function of $x$, $f(x)$:

$$f(x) = x - \frac{1}{x} - 2 \ln x.$$

It is easy to see that:

$$f(1) = 1 - 1 - 2 \ln 1 = 0,$$

and

$$f'(x) = 1 + \frac{1}{x^2} - \frac{2}{x} = \left(1 - \frac{1}{x} \right)^2 > 0, \quad \text{for } x > 1.$$

Therefore, $f(x)$ increases, and so $f(x) > 0$ for $x > 1$. Thus,

$$y(2\tau_m) > 0,$$

and hence it takes the ball longer to fall down than to go up.

### 3.2 Perturbative treatment of model (3.5)

The perturbative treatment of the above model can be motivated by two different observations.

First, we note that expression (3.10) is rather cumbersome. In practice, the air resistance is quite small, and so all we really need is the first-order correction to the equations of motion (3.8) without the air resistance. To obtain such a correction, we need to expand (3.10) keeping higher orders of $K$ than we did above when considering the limiting case $K \ll 1$. 
Second, note that the air resistance could have been given by a more complicated function of \(v\) than (3.2), in which case it would not be possible to obtain an exact analytical solution for the counterpart of model (3.5). Yet, as long as the air resistance is small, we could hope to find an approximate solution of that model as being a perturbation of the solution (3.8) without the air resistance.

Below we illustrate both approaches, starting with the first one mentioned above. Let us seek a representation of solution (3.10) as:

\[
y(\tau) = y^{(0)}(\tau) + Ky^{(1)}(\tau) + O(K^2),
\]

where \(y^{(0)}(\tau)\) is the resistance-free solution (3.8b) and \(y^{(1)}\) is the first-order correction to it. To find \(y^{(1)}\), we repeat the calculations done before Eq. (3.11), but keep one more power of \(K\):

\[
y(\tau) = \frac{1}{gK} \left(1 + Kv_0\right) \left[1 - \frac{1 - K\tau + \frac{K^2\tau^2}{2} - \frac{K^3\tau^3}{6} + O(K^4)}{K} - \tau\right]
\]
\[
= \frac{1}{gK} \left(1 + Kv_0\right) \left[\tau - \frac{K\tau^2}{2} + \frac{K^2\tau^3}{6} + O(K^3)\right] - \tau
\]
\[
= \frac{1}{gK} \left(\tau - \frac{K\tau^2}{2} + \frac{K^2\tau^3}{6} + O(K^3) + Kv_0\tau - \frac{K^2v_0\tau^2}{2} + O(K^3)\right) - \tau
\]
\[
= \frac{1}{gK} \left(K \left(v_0\tau - \frac{\tau^2}{2}\right) + K^2 \left(-\frac{v_0\tau^2}{2} + \frac{\tau^3}{6}\right) + O(K^3)\right)
\]
\[
= \frac{1}{g} \left(v_0\tau - \frac{\tau^2}{2}\right) + K \left(-\frac{v_0\tau^2}{2} + \frac{\tau^3}{6}\right) + O(K^2).
\]

Thus, we have obtained that

\[
y(\tau) = \frac{1}{g} \left(v_0\tau - \frac{\tau^2}{2}\right) + K \left(-\frac{v_0\tau^2}{2} + \frac{\tau^3}{6}\right) + O(K^2).
\] (3.12)

**Question:** When is this perturbative solution valid?

**Answer:** When the correction term is much smaller than the resistance-free term, i.e.

\[
\left|K \left(-\frac{v_0\tau^2}{2} + \frac{\tau^3}{6}\right)\right| \ll \left|v_0\tau - \frac{\tau^2}{2}\right|.
\] (3.13)

For (3.13) to hold, it suffices that each term on the l.h.s. be smaller than each term on the r.h.s. This occurs when (verify)

\[
Kv_0 \ll 1, \quad K\tau \ll 1, \quad \text{and} \quad K\tau^2 \ll v_0.
\] (3.14)

Let us show that all three of these inequalities are equivalent. Indeed, we are interested in the times when the ball is in the air, i.e. \(\tau \approx 2\tau_m\). One can alternatively write this as \(\tau \approx \tau_m\), where the symbol “\(\approx\)” means “equals in the order of magnitude sense”. E.g., \(1 \approx 2\) or \(1 \approx 3\), i.e. this new notation allows us to ignore a factor of order 2 in our formulæ.\(^1\) Next, we estimate \(\tau_m\) from (3.11). Using the Maclaurin series for \(\ln(1 + x)\):

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots, \quad \Rightarrow \quad \ln(1 + x) = x + O(x^2) \quad \text{for} \quad x \ll 1,
\]

\(^1\)A legitimate question to ask would be: Is \(1 \sim 10\)? An answer depends on particular circumstances. E.g., \(1 \sim 10\) if we compare both these numbers with 1000, but \(1 \not\sim 10\) if we compare them with 20.
we see that
\[ Kv_0 \ll 1 \Rightarrow \ln(1 + Kv_0) = Kv_0 + O(K^2). \]

From the last expression and (3.11), it follows that
\[ Kv_0 \ll 1 \Rightarrow \tau_m = v_0 + O(K), \]

which finally leads to
\[ \tau \sim v_0. \]

With the above estimate, it is now clear that all the three strong inequalities in (3.14) are equivalent.

Now, let us explore the second venue, described at the beginning of this section. Consider model (3.5) where the term \( K v \ll 1 \). Note that this is precisely the condition under which the perturbative solution is valid; however, in this case, it arises from purely physical consideration that the air resistance be small compared to the gravity.

Let us seek the solution \( v(\tau) \) in the form:
\[ v = v^{(0)} + K v^{(1)} + K^2 v^{(2)} + \ldots, \quad (3.15) \]
where \( v^{(0)} \), \( v^{(1)} \), \( v^{(2)} \), etc. do not depend on \( K \). Substituting (3.15) into (3.5) we obtain:
\[ \dot{v}^{(0)} + K \dot{v}^{(1)} + O(K^2) = -1 - K(v^{(0)} + K v^{(1)} + O(K^2)). \]

Let us now collect the terms at the like powers of \( K \):

at \( K^0 \):\n\[ \dot{v}^{(0)} = -1 \]

This is the equation for the resistance-free case, as expected. With the initial condition from (3.7), we have:
\[ v^{(0)} = v_0 - \tau \]
(which, of course, is (3.8a)). Next,

at \( K^1 \):
\[ \begin{align*}
\dot{v}^{(1)} &= -v^{(0)} \quad \text{(see the equation above)} \\
\dot{v}^{(1)} &= -v_0 + \tau \quad \Rightarrow \\
v^{(1)} &= v^{(1)}(\tau = 0) - v_0 \tau + \frac{\tau^2}{2} \\
&= -v_0 \tau + \frac{\tau^2}{2}.
\end{align*} \]

Here we have again used the initial condition (3.7), which implies that since \( v(\tau = 0) = v_0 \) and since we have taken \( v^{(0)}(0) = v_0 \), then \( v^{(n)}(0) = 0 \) for \( n = 1, 2, \ldots \). Substituting the expressions for \( v^{(0)} \) and \( v^{(1)} \) into (3.15), we find:
\[ v = (v_0 - \tau) - K \left( v_0 \tau - \frac{\tau^2}{2} \right) + O(K^2), \quad (3.16a) \]
and hence (using $y_0 = 0$ and omitting the $O(K^2)$-term):

$$y = \frac{1}{g} \left( \left( v_0 \tau - \frac{\tau^2}{2} \right) - K \left( \frac{v_0 \tau^2}{2} - \frac{\tau^3}{6} \right) \right).$$

(3.16b)

This is the same as (3.12), as it should be.

Thus, we have shown that the same perturbative solution can be obtained by two independent approaches: by Taylor-expanding the exact solution and by perturbatively solving the model, Eq. (3.5). In most practical cases, when the exact solution is not available, the second approach may be the only one that can give an approximate solution.

To conclude this section, let us find the perturbative expressions for the times of going up and down, and thereby confirm our earlier conclusion that going down takes longer. First, from (3.16a), we find the time of going up as the particular value of $\tau$ when $v = 0$:

$$0 = v_0 - \tau_m - K v_0 \tau_m + \frac{K \tau_m^2}{2},$$

(3.17)

where we have omitted the $O(K^2)$-term. This is a quadratic equation for $\tau$ and can be solved exactly. However, a much easier approach is to seek the solution $\tau_m$ in the form similar to (3.15):

$$\tau_m = \tau_m^{(0)} + K \tau_m^{(1)} + O(K^2).$$

(3.18)

Substituting (3.18) into (3.17), we obtain:

$$0 = v_0 - (\tau_m^{(0)} + K \tau_m^{(1)} + O(K^2)) - K v_0 (\tau_m^{(0)} + K \tau_m^{(1)} + O(K^2)) + \frac{K}{2} (\tau_m^{(0)} + K \tau_m^{(1)} + O(K^2))^2.$$

Collecting terms at like powers of $K$:

at $K^0$:

$$0 = v_0 - \tau_m^{(0)} \quad \Rightarrow \quad \tau_m^{(0)} = v_0.$$

at $K^1$:

$$0 = -\tau_m^{(1)} - v_0 \tau_m^{(0)} + \frac{(\tau_m^{(0)})^2}{2} \quad \Rightarrow \quad \tau_m^{(1)} = -v_0 \tau_m^{(0)} + \frac{(\tau_m^{(0)})^2}{2} = -\frac{v_0^2}{2}.$$

Thus,

$$\tau_m = v_0 - \frac{K v_0^2}{2} + O(K^2).$$

(3.19)

Verify that this agrees with the first two terms of the expansion of (3.11) when $K \ll 1$ (use the Maclaurin series stated after Eq. (3.14)).

Now let us use the same method to find the time, $\tau_h$, when the ball hits the ground. Substituting into (3.16b) with $y = 0$ an expansion

$$\tau_h = \tau_h^{(0)} + K \tau_h^{(1)} + O(K^2),$$

we find, omitting $O(K^2)$ terms:

$$0 = v_0 (\tau_h^{(0)} + K \tau_h^{(1)}) - \frac{1}{2} (\tau_h^{(0)} + K \tau_h^{(1)})^2 - K \left[ \frac{v_0}{2} (\tau_h^{(0)} + K \tau_h^{(1)})^2 - \frac{1}{6} (\tau_h^{(0)} + K \tau_h^{(1)})^3 \right].$$
Collecting the coefficients at like powers of $K$:

at $K^0$:

\[ 0 = v_0\tau_h^{(0)} - \frac{1}{2}(\tau_h^{(0)})^2 \implies \tau_h^{(0)} = 2v_0. \]

at $K^1$:

\[ 0 = v_0\tau_h^{(1)} - \frac{1}{2} \cdot 2\tau_h^{(0)} \tau_h^{(1)} - \left[ \frac{v_0}{2} \cdot (\tau_h^{(0)})^2 - \frac{1}{6}(\tau_h^{(0)})^3 \right]. \]

**IMPORTANT NOTE:** Although the original equation for $\tau_h$ was nonlinear (see (3.16b) with $y = 0$), the equation for the correction $\tau_h^{(1)}$ (and for all higher-order corrections $\tau_h^{(2)}$, $\tau_h^{(3)}$, etc., if we decide to find them) is linear, and hence can always be solved and yields a unique solution.

Continuing, from the above equation we have:

\[
\tau_h^{(1)}(v_0 - \tau_h^{(0)}) = \frac{v_0}{2}(\tau_h^{(0)})^2 - \frac{1}{6}(\tau_h^{(0)})^3;
\]

and, using the above expression for $\tau_h^{(0)}$:

\[
\tau_h^{(1)} = -\frac{2}{3}v_0^2
\]

(verify). Thus,

\[
\tau_h = 2v_0 - K \cdot \frac{2}{3}v_0^2 + O(K^2). \tag{3.20}
\]

From (3.19) and (3.20), the time required for the ball to go down is:

\[
\tau_h - \tau_m = 2v_0 - \frac{2K}{3}v_0^2 + O(K^2) - v_0 + \frac{K}{2}v_0^2 - O(K^2) = v_0 - K \cdot \frac{1}{6}v_0^2 + O(K^2). \tag{3.21}
\]

Comparing (3.21) with (3.19), we see that the time to go down is greater than the time to go up, as was proved in general in Section 3.1.

### 3.3 Model with quadratic air resistance

We will now follow the steps of Sections 3.1 and 3.2 to analyze the solution of the model with the air resistance force given by (3.3):

\[
\begin{array}{ll}
\nu > 0 & \dot{\nu} = -1 - K\nu^2, \tag{3.22a} \\
\nu < 0 & \dot{\nu} = -1 + K\nu^2 \tag{3.22b}
\end{array}
\]

(see (3.1) and (3.3)). (Note that physically, this model is not applicable to the motion of the ball, but the mathematical perturbation approach carries over to it without changes, and it is this approach that we intend to practice in this lecture.) I will go briefly over the main steps of the solution. You will be asked to supply the missing details in the homework. We have to analyze (3.22a) and (3.22b) separately, since these are different equations. Let us begin with (3.22a). The solution to (3.22a) and (3.7) is given by:

\[
\frac{\arctan(\sqrt{K}\nu)}{\sqrt{K}} = -\tau + \frac{\arctan(\sqrt{K}v_0)}{\sqrt{K}}, \tag{3.23a}
\]

...
\[ y = y_0 + \frac{1}{gK} \ln \left| \cos \left( \frac{\arctan(\sqrt{K}v_0) - \sqrt{K}\tau}{\cos(\arctan(\sqrt{K}v_0))} \right) \right|. \] (3.23b)

The exact time to reach the highest point is found from (3.23a):

\[ \tau_m = \frac{\arctan(\sqrt{K}v_0)}{\sqrt{K}}. \] (3.24)

As in Section 3.2, here our goal will be to obtain the perturbation-type solution for \( v, y, \) and \( \tau_m \) in two ways: by Taylor-expanding the exact solutions (3.23) and (3.24), and by perturbatively solving (3.22a).

Let us start with the Taylor expansion. It is, of course, possible to solve (3.23a) for \( v \) and Taylor-expand the answer, and also to Taylor-expand (3.23b). However, this is an awkward approach. It will be much easier to Taylor-expand (3.23a) without solving for \( v \), and then integrate the result with respect to \( \tau \) to obtain the approximate answer for \( y \). The reason why such an approach is easier is the same as why implicit differentiation is sometimes easier than explicit differentiation.

To obtain the approximate solution of (3.23a), substitute there expansion (3.15). Note that the entire arguments of both arctangents are small because \( K \ll 1 \). Then, use the Maclaurin expansion of \( \arctan x \):

\[ \arctan x = x - \frac{x^3}{3} + O(x^5), \quad x \ll 1 \]

to obtain that

\[ v = (v_0 - \tau) + K \left( -v_0^2 \tau + v_0 \tau^2 - \frac{\tau^3}{3} \right) + O(K^2). \] (3.25a)

In doing so, you should follow the lines of the derivation of Eqs. (3.19) and (3.20).

Integrate (3.25a) to obtain (with \( y_0 = 0 \)):

\[ y = \frac{1}{g} \left[ (v_0 \tau - \frac{\tau^2}{2}) + K \left( -\frac{v_0^2 \tau^2}{2} + \frac{v_0 \tau^3}{3} - \frac{\tau^4}{12} \right) \right] + O(K^2). \] (3.25b)

When are the perturbative expansions (3.25a) and (3.25b) valid? See a similar discussion after Eq. (3.12).

Finally, from (3.24), obtain:

\[ \tau_m = v_0 - K \frac{v_0^3}{3} + O(K^2). \] (3.26)

Thus, you have found the perturbative solution (3.25) and (3.26) by Taylor-expanding the exact solution (3.23) and (3.24).

Now follow the approach presented after Eq. (3.15) to re-obtain these results by the other method considered in Section 3.2. First, substitute expansion (3.15) into (3.22a) to re-obtain (3.25a). Then, to conclude the treatment of the upward motion of the ball, re-obtain (3.26) starting with (3.25a). Use expansion (3.18) for \( \tau_m \) and follow the approach presented after that equation.

Now, turn to the downward motion of the ball, described by Eq. (3.22b). As before, begin by finding its exact solution. In an implicit form, it is:

\[ \frac{1}{2\sqrt{K}} \ln \left| \frac{1 + \sqrt{K}v}{1 - \sqrt{K}v} \right| = -\tau + C, \] (3.27a)
where $C$ is the integration constant. At $\tau = \tau_m$ (i.e., when the ball is at the highest point of its trajectory), $v = 0$. From this condition, deduce the value of $C$, by setting $v = 0$ and $\tau = \tau_m$ in (3.27a).

Next, solve (3.27a) for $v$ to obtain:
\[
v = -\frac{1}{\sqrt{K}} \cdot \frac{1 - \exp(-2\sqrt{K}(\tau - C))}{1 + \exp(-2\sqrt{K}(\tau - C))}.
\]
(3.27b)

(Physically, what does the minus sign in front of this expression tell you?) To conclude solving for $v$, transform (3.27b) into:
\[
v = -\frac{1}{\sqrt{K}} \tanh(\sqrt{K}(\tau - C)).
\]
(3.28a)

To obtain $y(\tau)$, integrate (3.28a) and use the initial condition that at $\tau = \tau_m$, the ball is at its highest elevation. Let us denote this elevation $y_m$. Then, obtain from (3.28a) that
\[
y = y_m - \frac{1}{gK} \ln \left[ \cosh(\sqrt{K}(\tau - C)) \right].
\]
(3.28b)

Finally, find $y_m$ from (3.23b) (assuming that $y_0 = 0$). When you put all these results together, your answer for $y(\tau)$ should be equivalent to
\[
y = \frac{1}{gK} \ln \left( \frac{\sqrt{1 + K\nu_0^2}}{\cosh(\sqrt{K}(\tau - \arctan(\sqrt{K}\nu_0)))} \right).
\]
(3.29)

Interestingly enough, unlike for the model with the linear (in $v$) air resistance, for the model with the quadratic in $v$ air resistance, it is possible to find the analytic expression for the time when the ball hits the ground. By setting $y = 0$ in (3.29), show that this time, $\tau_h$, is given by:
\[
\tau_h = \tau_m + \frac{1}{\sqrt{K}} \ln \left( \sqrt{1 + K\nu_0^2} + \sqrt{K}\nu_0 \right).
\]
(3.30)

Here you need to make use of the identity
\[\text{arccosh } x = \ln (x + \sqrt{x^2 - 1}).\]

This concludes finding the exact solution for the going-down case.

From this point on, repeat the steps you did for the going-up case. That is, you will first obtain approximate solutions from the exact ones by Taylor-expanding the latter. To start, obtain the Taylor expansions of (3.27a), valid up to terms $O(K)$ (i.e., obtain the expression for the coefficient of the $O(K)$-term). This should be done by substituting expansion (3.15) into (3.27a) and then using the approach of obtaining Eqs. (3.19) and (3.20)². Next, integrate the expression for $v$ and obtain an $O(K)$-accurate expression for $y$. Then derive an $O(K)$-accurate expression for $\tau_h$ from (3.30). You should obtain the following results:
\[
v = \left( (\tau - C) - \frac{K}{3} (\tau - C)^3 + O(K^2) \right),
\]
(3.31a)

²It is actually easier to obtain (3.31a) by Taylor-expanding the explicit solution (3.28a) than the implicit solution (3.27a). The reason that you are asked to use the more difficult approach here is that it is more advanced and also more general. In particular, it does not rely on the possibility to solve an implicit equation, like (3.27a), for an explicit answer.
$$y = y_m - \frac{1}{g} \left( \frac{(\tau - C)^2}{2} - \frac{K(\tau - C)^4}{12} + O(K^2) \right), \quad (3.31b)$$

$$\tau_h = \tau_m + \left( v_0 - \frac{Kv_0^3}{6} + O(K\sqrt{K}) \right); \quad (3.31c)$$

a technical comment about the $O(K\sqrt{K})$-term in (3.31c) is found in the footnote\(^3\). In (3.31a) and (3.31b), substitute the expression for $C$ which you have found earlier. (You do \textit{not} need to Taylor-expand that expression for $C$ in powers of $K$.) When are the expansions (3.31) valid? This concludes the step of obtaining approximate solutions from the exact solutions (3.27)–(3.30).

Finally, re-obtain the approximate solutions (3.31) starting from the differential equation (3.22b). To begin, re-obtain (3.31a) by substituting expansion (3.15) into (3.22b). Once you have obtained (3.31a), it can be integrated to yield (3.31b). You do not need to do this integration here because you have already done it above when obtaining (3.31b) for the first time. Now, to conclude, use the expansion for $(\tau_h - \tau_m)$ similar to the one found after Eq. (3.19), i.e.:

$$\tau_h - \tau_m = \Delta\tau^{(0)} + K\Delta\tau^{(1)} + O(K^2), \quad (3.32)$$

to re-obtain (3.31c) from (3.31b). Do this in two steps. First, express the answer in terms of $y_m$ \textit{without} expanding $y_m$ in powers of $K$. Your answer should look like (3.32) where $\Delta\tau^{(0)}$ and $\Delta\tau^{(1)}$ may depend on $y_m$. Then, expand the expression for $y_m$ in powers of $K$ and substitute it into your expression for $(\tau_h - \tau_m)$ to make it look like (3.31c). In so doing, you should use a property of logarithms before expanding $\ln\sqrt{1 + x}$ in a Maclaurin series.

\(^3\)It is possible to show that it is actually $O(K^2)$, but this is more difficult to do; so just show that this term is no greater than $O(K\sqrt{K})$. 