2 Kepler’s Laws

2.1 Some history

2.1.1 End of 16th century

Tycho Brahe (pronounced [taıkou brachi:]), a wealthy Danish nobleman, had a passion for observation of stars. He was the director of the Prague Observatory. Since he was very rich, he could afford to construct gigantic observational instruments, with which he collected a great number of new data with precision unknown before him. (Note: a telescope was invented in 1610, nine years after Brahe’s death.)

2.1.2 Beginning of 17th century

Johannes Kepler, originally from Germany, served as an assistant to Tycho Brahe at the Prague Observatory. He succeeded Brahe as the director and inherited a vast collection of observational data.

Kepler’s initial ambitious goal was to describe precisely the shape of the orbit of Mars. First, he computed its period of revolution (about the Sun): \( T_{\text{Mars}} = 687 \text{(Earth) days} \approx 1.88 \text{ Earth years} \). Then, knowing the distance from the Earth to the Sun and using other astronomical data, Kepler, after many unsuccessful attempts, determined that the Mars orbit is an ellipse and the Sun is one of its foci. Moreover, he found that Mars’ motion was not uniform: the farther it is from the Sun, the slower it moves. These discoveries were heretical to Kepler because they indicated that the Universe was not perfectly symmetric, something Kepler could not believe for a long time. Indeed, why is the Sun in one foci and not in the other? Yet, the data were too compelling for Kepler not to accept his new model of the elliptical motion. He then extended his model to other planets. By 1609, he formulated his first 2 laws:

(i) Each planet moves on an ellipse with the Sun at one focus.

(ii)

\[ \text{faster} \]

\[ \text{slower} \]

\[ \text{Sun} \]

For each planet, the line from the Sun to the planet sweeps out equal areas in equal times.

It took Kepler 10 more years to formulate his 3rd law:

(iii) The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit:

\[ \frac{T^2}{a^3} = \text{const (same for all planets)} \]
How could Kepler find the exponents 2 and 3? He used recently invented logarithms! Namely, suppose that two given lists of data, $x$ and $y$ are related by

$$y = k \cdot x^n.$$ 

How can we find $k$ and $n$? Answer: Take the logarithm of the above equation to obtain

$$\log y = \log k + n \log x.$$ 

This equation defines a linear relationship (a straight line) between $\log y$ and $\log x$; its slope is $n$ and the intercept is $\log k$.

At home you will be asked to verify Kepler’s Third law given the data for the planets from Mercury to Pluto.

### 2.1.3 End of 17th century

In 1680, Robert Hooke wrote a letter to Newton in which he, among other things, wrote about his two new hypotheses:

- the gravitational force of attraction between two bodies falls off as $1/r^2$, where $r$ is the distance between them;
- bodies in such a field move along ellipse-looking orbits. (Hooke’s scientific ethics did not allow him to say “ellipses” because he did not have any rigorous proof that they were not some other kind of ovals.)

This letter might have set Newton on the path of deriving the inverse-square law for the gravitational force from the three laws of Kepler, and, vice versa, deriving the Kepler’s laws from the Universal Gravitation law and Newton’s Second law of mechanics ($m\ddot{a} = F$).

However, Newton never acknowledged the letter from Hooke. His argument to their common friend Edmund Halley about not doing so was that all Hooke had done was to get an intuitive idea, while it was Newton who rigorously proved it. Moreover, when Newton became the President of the Royal Society, he ordered to destroy not only all works and apparatuses of Hooke (who died by then; Hooke used to be the Curator of the Royal Society — a person responsible for demonstrations of new experiments at the Society’s meetings), but even all portraits of Hooke!

### 2.2 Newton’s law of Universal Gravitation

Although Newton knew Calculus, he did not use it in his famous book "Principia Mathematica", in which he derived the Universal Gravitation law and the Kepler’s laws. Instead, he used elementary geometrical methods. An approach similar to Newton’s is described in an article by Hall & Higson, posted online. Here we will use the modern approach based on Calculus, but will also try to reconstruct the main steps that Newton had to follow.

In this section, we describe the two steps that Newton had to accomplish to derive his Universal Law of Gravitation (assuming that he knew his Second law, $m\ddot{a} = F$) from Kepler’s laws. Namely, we will use Kepler’s laws to show that:

1. the gravitational force is central;
2. its magnitude follows the $1/r^2$ dependence.
2.2.1 Showing that the gravitational force is central

The Second Kepler’s law says that the planet sweeps out the same area \( \Delta A \) during an interval \( \Delta t \) anywhere along its orbit. In general,

\[
\Delta A \simeq \frac{1}{2} r^2 \Delta \theta ,
\]

since \( r \simeq \text{const} \) when \( \Delta \theta \) is small. Dividing this expression by \( \Delta t \) yields

\[
\frac{\Delta A}{\Delta t} \simeq \frac{1}{2} r^2 \frac{\Delta \theta}{\Delta t} , \quad \text{or} \quad \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} .
\]

Let us introduce a notation that will be used extensively in this course:

\[
\frac{dq}{dt} = \dot{q} , \quad \text{for any quantity } q .
\]

Thus, the Second Kepler’s law can be written as:

\[
\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \text{const} = \frac{1}{2} c . \quad (2.1)
\]

Now,

\[
r^2 \dot{\theta} = xy - \dot{x}y . \quad (2.2)
\]

Indeed,

\[
x = r \cos \theta ; \quad y = r \sin \theta ; \quad \dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta ; \quad \dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta . \quad (2.3)
\]

Substituting these expressions into the r.h.s. of (2.2), we obtain its l.h.s. (verify). Since this l.h.s. = const, then

\[
\frac{d}{dt}(xy - \dot{x}y) = 0 \quad \Rightarrow \quad xy - \dot{x}y = 0 \quad (\text{verify}) . \quad (2.4)
\]

Next, (2.4) can be used to show that \( \vec{r} \times \ddot{\vec{r}} = \vec{0} \). Indeed, if \( \vec{r} = (x, y, 0) \), \( \ddot{\vec{r}} = (\ddot{x}, \ddot{y}, 0) \), then \( \vec{r} \times \ddot{\vec{r}} = \vec{i} \cdot 0 + \vec{j} \cdot 0 + \vec{k} (x \ddot{y} - \ddot{x}y) = \vec{0} \) (verify), which means \( \vec{r} \parallel \ddot{\vec{r}} \), or equivalently,

\[
\ddot{\vec{r}} = f(r, \theta) \vec{r} \quad (2.5a)
\]

for some scalar function \( f(r, \theta) \). Finally,

\[
m \ddot{\vec{r}} = \vec{F} \quad \Rightarrow \quad m \vec{\ddot{r}} = \vec{F} \quad \Rightarrow \quad \vec{F} = mf(r, \theta) \vec{r} , \quad (2.5b)
\]

i.e. the force is collinear with \( \vec{r} \) and hence is central.
### 2.2.2 Showing that $f(r, \theta) = \text{const}/r^2$

The planets’ orbits are, in fact, nearly circular (and this was known to Kepler and Newton). Let us assume for now that the orbits are exactly circular. Then, using the polar equation of a circle, we have:

\[
\vec{r} = \langle r \cos \theta, r \sin \theta \rangle
\]

\[
r = \text{const}, \quad \theta = \frac{2\pi t}{T},
\]

where $T$ is the period of the planet’s revolution about the Sun. By differentiating the above expression, one finds (verify):

\[
\ddot{\vec{r}} = -\left(\frac{2\pi}{T}\right)^2 \vec{r}.
\]

Comparing this with Eq. (2.5a), we have:

\[
f(r, \theta) = -\frac{4\pi^2}{T^2}.
\]

But by Kepler’s Third law (applied to circular orbits), $T^2 = \text{const} \cdot r^3$. Hence $f(r, \theta) \sim 1/r^3$ and therefore

\[
\vec{F} = -\frac{\text{const}}{r^3} \vec{r}, \quad \text{or} \quad \lVert \vec{F} \rVert \sim \frac{1}{r^2}.
\]  \hspace{1cm} (2.6)

**Side note:** Kepler also thought of the inverse-square law for the gravitation, but rejected the idea. His reasoning was approximately the following.

Kepler thought of the gravitational field emanated by a body as being similar to light. The intensity of light away from its source falls off as $1/r^2$:

\[
\frac{\text{source strength}}{\text{surface area}} = \frac{\text{const}}{4\pi r^2}.
\]

So, it then would be natural that the intensity of the emanated gravitational field also falls off as $1/r^2$.

However, if this model were correct, then it would contradict to what happens during the solar eclipse. In this event, the Moon blocks the Sun’s light (and, as Kepler thought, gravitation) from reaching the Earth, and hence during solar eclipses, the motion of the Earth should experience large perturbations. Since this was never observed, Kepler rejected the inverse-square law for gravitation.

The reason this theory (and its rejection by Kepler) became known to us is that Kepler in his publications did not “try to cover his traces” (unlike most scientists). He wrote about his conjectures, failures, successes, errors, insights, etc. with great frankness. The downside of this style is that when many competing ideas are presented in a paper, most readers are lost. So it took Newton’s genius to separate the wheat from the chaff and discern the importance of Kepler’s three laws.
2.3 Auxiliary formulae from vector Calculus

We will now proceed in the reverse direction: We will use Newton’s Law of Universal Gravitation to derive the three Laws of Kepler. For that, we need to remind or establish four auxiliary facts.

**Fact 1** For any three vectors $\vec{a}, \vec{b}, \vec{c}$:

$$\left( \vec{a} \times \vec{b} \right) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}.$$  \hspace{1cm} (2.7)

This formula can be established by direct calculation. However, there is an easier (and more mathematically literate) method. Let

$$\vec{a} = \sum_{l=1}^{3} a_l \vec{e}_l, \quad \vec{b} = \sum_{m=1}^{3} b_m \vec{e}_m, \quad \vec{c} = \sum_{n=1}^{3} c_n \vec{e}_n, \quad \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \{\vec{i}, \vec{j}, \vec{k}\},$$

where $a_l$ are the usual cartesian coordinates of $\vec{a}$, etc. Then the l.h.s of (2.7) can be rewritten as:

$$\sum_{n=1}^{3} \sum_{m=1}^{3} \sum_{l=1}^{3} a_l b_m c_n \cdot (\vec{e}_l \times \vec{e}_m) \times \vec{e}_n.$$  

Therefore, since $a_l, b_m, c_n$ can be arbitrary, it suffices to verify that (2.7) holds for the unit coordinate vectors $\{\vec{i}, \vec{j}, \vec{k}\}$ in different permutations. (Note that out of 27 such permutations, 9 with $l = m$ are immediately seen to be zeros.)

**Fact 2**

Recall that the *triple product* is related to the volume of a parallelepiped whose sides are $\vec{a}, \vec{b}, \vec{c}$:

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|.$$  

Also,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}.$$  \hspace{1cm} (2.8)

**Fact 3** Suppose $\vec{r}(t)$, and hence $\dot{\vec{r}}(t)$, are known. We can explicitly separate the length and direction of this vector:

$$\vec{r} \equiv r \cdot \left( \frac{\vec{r}}{r} \right),$$

where $r$ is the length and $(\vec{r}/r)$ is the unit vector along $\vec{r}$. Then what are $\dot{r}$ and $(\vec{r}/r)'$? First, let us find $\dot{r}$:

$$r^2 = (\vec{r} \cdot \vec{r}) \quad \Rightarrow \quad (r^2)' = (\dot{\vec{r}} \cdot \vec{r}) + (\vec{r} \cdot \dot{\vec{r}}) \quad \Rightarrow \quad 2r\dot{r} = 2(\dot{\vec{r}} \cdot \vec{r}) \quad \Rightarrow \quad \dot{r} = \frac{(\dot{\vec{r}} \cdot \vec{r})}{r}.$$  \hspace{1cm} (2.9)
Then using the last formula and the Product Rule
\[ \vec{r} = \left( r \left( \frac{\vec{r}}{r} \right) \right)' = \dot{r} \left( \frac{\vec{r}}{r} \right) + r \left( \frac{\vec{r}}{r} \right) , \]
one finds (verify):
\[ \left( \frac{\vec{r}}{r} \right)' = (\vec{r} \cdot \vec{r}) \dot{r} - (\dot{\vec{r}} \cdot \vec{r}) \frac{\vec{r}}{r^3} . \]
Finally, using Eq. (2.7), we obtain a formula that we will use in the next section:
\[ \left( \frac{\vec{r}}{r} \right)' = \frac{(\vec{r} \times \dot{\vec{r}}) \times \vec{r}}{r^3} . \] (2.10)

**Fact 4** Conic sections in polar coordinates.

**Definition** A conic section is a planar curve whose points \( P \) satisfy the following property:
\[ |OP| = e|PQ| \]
where the notations are as shown in the figure and \( e \) is a fixed number called *eccentricity*.

On p. 71 of the book “Inverse Problems” by C.W. Groetsch (MAA 1999), it is shown that the polar equation of a conic is
\[ r = \frac{eD}{1 + e \cos \theta} , \] (2.11)
where \( D = |OS| \) is the (given) distance between the focus and the directrix.

In a homework problem you will show that Eq. (2.11) reduces to more familiar Cartesian forms of three different conic sections depending on the value of \( e \).

With this background information, we are ready to derive the three Kepler’s laws from Newton’s Law of Universal Gravitation (Eq. (2.6)) and the Second law of motion, \( m\ddot{\vec{r}} = \vec{F} \).

### 2.4 Derivation of Kepler’s laws from Newton’s laws

We begin by restating Newton’s Law of Universal Gravitation, (2.6), and the Second law of motion as one differential equation:
\[ \ddot{\vec{r}} = -\frac{N}{r^3} \vec{r} , \] (2.12)
where \( N \) is a proportionality constant. From this, we will derive the first two Kepler’s laws. In fact, it is convenient to derive Kepler’s Second law before the First, and so below we will state them in this order and also will slightly modify the statement of the Second law.

- **Kepler’s Second law, slightly modified**
A planet’s trajectory around the Sun is a planar curve (i.e., lies in one plane), and the radius-vector from the Sun to the planet sweeps out equal areas in equal time intervals.
The aforementioned modification is contained in the first clause of the previous sentence. It is actually borrowed from the First law, where one does not specify the shape of that planar curve. Then the First law specifies what that shape is:

- **Kepler’s First law**
  The trajectory of each planet is an ellipse with the Sun at one of its foci.

**Derivation of Kepler’s Second law**

Since we want to prove a statement about the area of a sector swept by the radius-vector of a planet, we begin by establishing a formula for that area. It (the area) is shown in the figure on the left and is seen to be approximately equal to the area of the triangle with one side \( \vec{r} \) and the base \( \vec{r'} \). From vector Calculus, you should recall that this area equals one half of \( |\vec{r} \times \vec{r'}\Delta t| \). Thus, the area swept by the radius-vector per time \( \Delta t \) is proportional to \( |\vec{r} \times \vec{r'}| \), and hence to demonstrate the second clause in the modified Second Kepler’s law stated above, we need to show that

\[
|\vec{r} \times \vec{r'}| = \text{const.} \tag{2.13}
\]

Let us now use the first clause of that law to show that we need the vector \( \vec{r} \times \vec{r'} \), not just its length, to stay constant. Indeed, if the trajectory lies in a plane, one can define a normal vector to that plane, and the direction of this normal vector stays constant at all times. This normal vector, by definition, is perpendicular to any vector in the plane of motion and hence has the direction of \( \vec{r} \times \vec{r'} \), by a property of the cross product. Hence the direction of \( \vec{r} \times \vec{r'} \) must stay constant. Along with (2.13) this implies that we want to show that

\[
\vec{r} \times \vec{r'} = \text{const} \equiv \vec{c} \tag{2.14}
\]

This is shown as follows. Take the cross-product of Eq. (2.12) with \( \vec{r} \) and use a property of the cross product to see that

\[
\vec{r} \times \vec{r'} = 0 \tag{2.15}
\]

Next, using the Product Rule, one obtains (verify):

\[
\vec{r} \times \vec{r'} = (\vec{r} \times \vec{r'})^\prime - (\vec{r} \times \dot{\vec{r'}}) \quad \Rightarrow \quad (\vec{r} \times \vec{r'})^\prime = 0 \tag{2.16}
\]

By integrating the last equation, one obtains (2.14).
Let us note in passing that
\[
\vec{c} = \vec{r} \times \dot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = \vec{k}(\dot{x}y - \dot{y}x).
\]
Therefore, $|\vec{c}| = |x\dot{y} - \dot{x}y|$, which has been shown to be a constant by a different method: see (2.1) and (2.2).

**Derivation of Kepler’s First law**

Our goal is now to show that a solution of the differential equation (2.12) is an ellipse. If our differential equation had the form \(\dot{\vec{r}} = \vec{f}(t)\), one could integrate it once to get the solution:
\[
\vec{r}(t) = \int_{t_0}^{t} \vec{f}(t') dt' + \vec{r}_0.
\]
Similarly, to solve \(\ddot{\vec{r}} = \vec{f}(t)\), one would integrate twice. However, in Eq. (2.12), the r.h.s. itself depends on the unknown \(\vec{r}(t)\). Moreover, the solution we are looking for does not have the form \(\vec{r} = \vec{r}(t)\) but, instead, has the form \(r = r(\theta)\): see (2.11). So we have to devise a trick to find that solution. This trick has two steps.

**Step 1:** Integrate (2.12) once to get an equation for \(\dot{\vec{r}}\)

In Section 2.3 we showed that (see (2.10)) for any law of motion,
\[
\left( \frac{\vec{r}}{r} \right)' = \frac{\vec{c} \times \vec{r}}{r^3},
\]
where \(\vec{c}\) is defined in (2.14). The r.h.s. of Eq. (2.17a) resembles the r.h.s. of the Newton’s equation (2.12) if we multiply the latter by \(\vec{c}\):
\[
\vec{c} \times \ddot{\vec{r}} = -\frac{N}{r^3} (\vec{c} \times \vec{r}).
\]
Comparing (2.17a) and (2.17b) and using the fact that \(\vec{c} = \text{const}\), we obtain:
\[
-N \left( \frac{\vec{r}}{r} \right)' = (\vec{c} \times \dot{\vec{r}}).
\]
Integrating this once and using the identity \(\vec{c} \times \dot{\vec{r}} = -\dot{\vec{r}} \times \vec{c}\), we get
\[
\dot{\vec{r}} \times \vec{c} = N \left( \vec{c} + \frac{\vec{r}}{r} \right),
\]
where \(\vec{c}\) is the integration constant.

Note that \(\vec{c}\) must be perpendicular to \(\vec{c}\): This follows from (2.18) (verify) once you recall that \(\vec{r} \perp \vec{c}\) (see (2.16)) and also \((\dot{\vec{r}} \times \vec{c}) \perp \vec{c}\). Therefore, according to the discussion after Eq. (2.16), \(\vec{c}\) lies in the plane of motion.

**Step 2:** Eliminate \(\dot{\vec{r}}\) from (2.18)

To complete the derivation, we take the dot product of (2.18) with \(\vec{r}\):
\[
(\vec{r} \times \dot{\vec{r}}) \cdot \vec{c} = N(\vec{r} \cdot \vec{c} + r),
\]
where on the l.h.s. we have used (2.8). Using the definition of $\vec{c}$ from (2.16), we rewrite the last equation as
\[
c^2 = N(\vec{r} \cdot \vec{c} + r).
\] (2.19)

Also, since $\vec{e}$ lies in the plane of motion, we can write:
\[
\vec{r} \cdot \vec{e} \equiv r \cdot e \cdot \cos \theta.
\]

Finally, from the last equation and (2.19), we get (verify):
\[
r = \frac{\left( \frac{c^2}{Ne} \right) e}{1 + e \cos \theta},
\]
which is the Equation (2.11) of a conic section.

**Derivation of Kepler’s Third law**

You will do it in a homework problem following the hints provided there.

As a note about notations, recall that Kepler’s Third law is:
\[
T^2 : a^3 = \text{const.}
\] (2.21)

Then, if $T$ is measured in (Earth) years and $a$ is measured in Astronomical units (1 A.U. = half of the major axis of the Earth orbit, or approximately the average distance from Sun to Earth), the Third law takes on a simple form:
\[
T^2 : a^3 = 1
\]

(why?). For example, the Halley’s comet (last seen in 1986, eccentricity 0.97) has a semi-major axis $a = 18.1$ A.U. When will it be seen again?