16. Poker

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Definition of Poker Hands

Each poker hand consists of five cards with the following properties:

1. A royal flush consists of the A K Q J 10 in the same suit: e.g., A♥, K♥, Q♥, J♥ 10♥.
2. A straight flush consists of a contiguous sequence in the same suit: e.g., 3♥, 4♥, 5♥, 6♥, 7♥. Thus, a Royal Flush is the highest ranking Straight Flush. Note that Aces can be high or low, but not both. So a Q♠, K♠, A♠, 2♠, 3♠ is not a straight flush.
3. Four of a kind contains four cards of the same rank: e.g., A♥, A♠, A♦, A♣, 2♥.
4. A full house contains three-of-a-kind as well as a pair: e.g., K♥, K♠, K♦, Q♦, Q♣.
5. A flush consists of five cards in the same suit that do not form a straight flush: e.g., A♣, 2♣, 3♣, Q♣, K♣.
6. A straight consists of five cards in a contiguous sequence, but not all in the same suit: e.g., 4♠, 5♠, 6♦, 7♥, 8♠.
7. A three-of-a-kind contains only three cards of the same rank, e.g. 7♠, 7♥, 7♣, K♦, J♣.
8. A two pair contains two pairs of cards of the same rank: e.g., 8♦, 8♠, 6♥, 6♣, Q♦.
9. A pair contains only a single pair of the same rank: e.g., J♠, J♣, A♥, 2♥, 3♥.
10. A high card: None of the above: e.g., Q♣, 10♥, 8♦, 7♦, 2♠.
Number of different hands

How many different poker hands are there? You might try to count them individually:

1. A♦, 2♣, 3♠, 4♦, 5♠,
2. A♠, 2♣, 3♠, 4♦, 6♠,
3. A♠, 2♣, 3♠, 4♦, 7♦,
4. A♠, 2♣, 3♠, 4♦, 8♠,

But since there are so many hands (as it turns out), this approach takes too long.

Instead, we will study the more general problem of combinations: how many ways can one form a subset of \( k = 5 \) elements, using different members from a set that contains \( n = 52 \) elements. This number, we will call “\( n \) choose \( k \),” or “52 choose 5.” It is often written in two different ways:

\[
\binom{52}{5} \quad \text{or} \quad C_5^{52}.
\]

(Note that, in general, the larger number appears above the smaller.)
Counting combinations

The following table represents the first poker hand in our previous list: A♥, 2♥, 3♥, 4♥, 5♥:

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<tr>
<th>A</th>
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In the above, a Y indicates that the card in that column is included in the hand, while an N indicates that it is not. Alternatively, the poker hand A♥, 2♥, 3♥, 4♥, 5♥ is represented by the 52-letter word,

YYYYYNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNN.

Note that every anagram of 5 Ys and 47 Ns corresponds to a different poker hand. In addition, for every poker hand there exists a corresponding anagram of 5 Ys and 47 Ns. (In set theory, this is called a one-to-one correspondence.)

Thus the number of poker hands equals the number of anagrams (i.e., unique permutations) of a 52 letter word that contains 5 Ys and 47 Ns, which is

$$\frac{52!}{5! \cdot 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960.$$
Evaluating $P$(Royal Flush)

More generally,

$$C^n_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Thus, the number of poker hands, “52 choose 5” is

$$\binom{52}{5} = 2,598,960.$$ 

In *five-card stud poker*, a player is eventually dealt five cards. (There are no additional draws or discards.) If the dealer is fair, then every one of these hands is equally likely to be dealt.

Under this assumption, the probability of receiving a particular kind of hand, (e.g., a Royal Flush) equals the ratio of the number of ways of being dealt that kind of hand to 2,598,960.

Since there are only four Royal Flushes:

- $\spadesuit A \spadesuit K \spadesuit Q \spadesuit J \spadesuit 10 \spadesuit$,  
- $\heartsuit A \heartsuit K \heartsuit Q \heartsuit 10 \heartsuit$,  
- $\diamondsuit A \diamondsuit K \diamondsuit Q \diamondsuit J \diamondsuit 10 \diamondsuit$,  
- $\clubsuit A \clubsuit K \clubsuit Q \clubsuit J \clubsuit 10 \clubsuit$,  

Evaluating \( P(\text{Straight Flush}) \)

To estimate the probability of being dealt a straight flush in five-card stud, we first need to enumerate the number of ways of being dealt a straight flush. Since aces can be either high or low, there are ten (10) straights in each suit:

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Since there are four (4) suits, which can be chosen independently from the ranks,

\[
\text{number of straight flushes} = 10 \times 4 = 40.
\]

This product is a result of the important multiplication principle:

*If the elements of a particular set can be constructed using a sequence of \( k \) steps, such that the number of outcomes of each step, \( n_1, n_2, \ldots, n_k \) are fixed, then the size of the set equals*

\[
\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k,
\]

*the product of the number of outcomes for every step.*

Thus, the number of straight flushes equals \( n_1 \times n_2 \), where \( n_1 = 10 \), the number of rank sequences, and \( n_2 = 4 \), the number of suits.
Estimating $P(\text{four-of-a-kind})$

Applying the multiplication principle, a hand that counts as a four of a kind can be constructed in two steps:

1. Select the rank of the four-of-a-kind, for which $n_1 = 13$;
2. Select the fifth card, for which $n_2 = 48$, the number of cards that remain after the first step.

Thus,

$$\text{# of four-of-a-kinds} = n_1 \times n_2 = 13 \times 48 = 624.$$ 

Consequently,

$$P(\text{four-of-a-kind}) = \frac{624}{2,598,960} = \frac{1}{4165} \approx 2.401 \times 10^{-4}.$$
Estimating \( P(\text{full-house}) \)

In enumerating the hands that constitute a full-house we apply the multiplication principle with a four step process: let,

\[
\begin{align*}
    n_1 &= \text{the number of ways to select the rank of the three-of-a-kind} = 13; \\
    n_2 &= \text{the number of ways to select the suits of the three-of-a-kind,} \\
    &= \binom{4}{3} = 4; \\
    n_3 &= \text{the number of ways to select the rank of the pair} = 12; \\
    n_4 &= \text{the number of ways to select the suits of the pair} = \binom{4}{2} = 6.
\end{align*}
\]

Thus, the number of unique hands that constitute a full house is

\[
    n_1 \times n_2 \times n_3 \times n_4 = 13 \times 4 \times 12 \times 6 = 3,744,
\]

and,

\[
    P(\text{full-house}) = \frac{3,744}{2,598,960} \approx 1.441 \times 10^{-3}.
\]
Four flushes

A hand that contains four cards of the same suit, with the remaining card in a different suit, that doesn’t contain a pair or a straight is called a four flush. In poker, it is usually considered as a special kind of high card hand. What is the probability of being dealt a four-flush?

We apply the multiplication principle using a four-step process. Let,

\[ n_1 = \text{the number of possible suits of the four flush} = 4, \]
\[ n_2 = \text{the number of possible suits for the remaining card} = 3, \]
\[ n_3 = \text{the number of possible ranks of the four flush cards} = \binom{13}{4} = 715, \]
\[ n_4 = \text{the number of possible ranks for the remaining card} = 9. \]

Note that the above choice of \( n_4 \) prevents the occurrence of a pair. Nevertheless, the product \( n_1 \times n_2 \times n_3 \times n_4 \) exceeds the correct value, because it also includes the number of straights that can be formed using four cards in one suit, and the remaining card in another suit, e.g., 2♥, 3♥, 4♥, 5♥, 6♠.
We shall now use the multiplication principle to obtain the number of such straights, again with a similar four-step process. Let

\[
\begin{align*}
m_1 &= \text{the number of possible suits of the four flush} = 4, \\
m_2 &= \text{the number of possible suits for the remaining card} = 3, \\
m_3 &= \text{the number of rank sequences of a (five) straight} = 10, \\
m_4 &= \text{the position of the remaining card in the (five) straight} = 5.
\end{align*}
\]

Thus,

\[
\text{# of four flushes} = n_1n_2n_3n_4 - m_1m_2m_3m_4, \\
= 4 \times 3 \times 715 \times 9 - 4 \times 3 \times 10 \times 5 = 76,620,
\]

and,

\[
P(\text{four flush}) = \frac{76,620}{2,598,960} \approx 2.948 \times 10^{-2}.
\]