15. Games of Chance

Robert Snapp
snapp@cs.uvm.edu

Department of Computer Science
University of Vermont
A Little History

Coin Tossing
- Pascal’s Triangle
- Binomial Coefficients

The Game of Craps

Probability Theory
- Independence
- Geometric Series
- Expected Reward

Lotteries
The History of Games of Chance

Games of chance are rooted in divination.

1. In *belomancy* arrows in a quiver are labeled according to different military options. The first arrow chosen from the quiver, or the one that flew the farthest, indicated the selection. A biblical reference to belomancy occurs in the book of Ezekial 21:21, “He shake his arrows.”

2. The Korean game of chance, *Nyout*, is used for divination on the 15th day of the first month of the year (Hargrave, 1930).

3. Labeled yarrow sticks are thrown at random, and read for prophesy according to the Chinese *I Ching*.

4. Herodotus (*The Histories*) describes Scythian soothsayers, who spread bundles of sticks on the ground, for divination.

5. Suetonius (*Divus Julius, XXXII*) asserts that Julius Ceasar uttered “*Jacta alea est*,” or “The die is cast,” as he crossed the Rubicon in 49 B.C. on his return to Rome.

6. Tarot cards, used since the middle ages in Europe for predicting fortunes.

7. Fortune cookies!
Kleromancy

Sheep knucklebones, or *astragali*, usually land in one of four different positions, which are assigned the values 1, 3, 4, and 6. In Ancient Greece five were thrown together, and the resulting sum was identified on a table of prophecies. From the oracular tables of Adada and Limyra:

- You will have a difficult harvest season.
- You have this righteous judgement from the gods.
- Succeeding, friend, you will fulfill a golden oracle.
- Having done something carelessly, you will blame the Gods.
- The affair holds a noble understanding.
- You will have a parting from the companions now around you.
- Phoibos [Apollo] speaks plainly, “Stay, friend.”
- You will go on more easily if you wait a short time.”
- Completing many contests, you will seize the crown.
- There are not crops to be reaped that were not sown.
- There is nor fruit to take from a withered shoot.
- The strife-bearing gift fulfills the oracle.
- It is necessary to labor, but the change will be admirable.

The four final positions of *astragali*; From left to right, one, three, four, and six points. From Jean-Marie L'Hôte, *Histoire des Jeux de Société*, Flammarion, Paris, 1994, p. 566.

Astragali

The Game of Jacks?

Two girls playing jacks with knucklebones in the lower-left corner of Pieter Bruegel's *Kinderspiele*.
Cowrie Shell Dice: India, 19th century

Stick Dice: India, 19th century

Cubic Dice: India, 19th century

Many sided dice: China, Han dynasty, 206 BCE– 220 CE

Coin Tossing

A coin offers two outcomes: $H$ or $T$. (One may think of a coin as a two-sided die.)

Let’s now consider a sequence of coin tosses. For example, if a coin is tossed three times, we might see $HTT$ (indicating that the first toss lands heads, and the subsequent two tosses land tails). Other possibilities are $THT$, $TTT$, etc. How many different sequences of length three exist?

This question can be answered by the multiplication principle. Let $n_1$ denote the number of different possible outcomes for the first toss; $n_2$, for the second toss; and $n_3$ for the third. Here,

$$n_1 = n_2 = n_3 = 2. \quad \text{(Why?)}$$

Thus by the multiplication principle, the number of different outcomes equals

$$N = n_1 \times n_2 \times n_3 = 2 \times 2 \times 2 = 2^3 = 8.$$

How many different outcomes exist for a sequence of four tosses? How about five tosses? Six tosses? Etc.
Coin Tossing (cont.)

It is not difficult to list all of the eight possible sequences:

$$TTT$$
$$HTT, \quad THT, \quad TTH$$
$$HHT, \quad HTH, \quad THH$$
$$HHH.$$ 

Note that there is only one sequence that contain exactly zero heads (or three tails), three sequences that contain exactly one head, three sequences that contain exactly two heads, and only one that contains exactly three heads.

How do these numbers relate to the number of different anagrams that can be formed with each set of letters? How many anagrams can be formed from the letters in the word $TTT$? How many for $HTT$? etc.

How many different possible sequences exist for four consecutive coin tosses? How many of these contain exactly 0 heads? How many contain 1 head? How many 2 heads? 3 heads? 4 heads?

Binary Trees

In a binary tree each node has two distinct branches: a left branch, and a right branch. (Some nodes will lack one or both of these, e.g., the leaf nodes). Let the left branch correspond to $H$ and the right branch correspond to $T$. The level correspond to the location of that toss in the sequence. A binary tree of depth 3 displays every possible outcome of three consecutive coin tosses.
Coin Tossing (cont.)

If the coin is fair, then roughly one half of a batch of tosses lands $H$, the remaining tosses being $T$. We can use probability to express and quantify the regularity in random events.

Thus, for a fair coin: $P\{H\} = P\{T\} = 1/2$.

What is the probability of getting exactly one heads in three coin tosses?
The number of leaf nodes is 32. The probability of each (for a fair coin) is $1/32$. In general, a complete binary tree of depth $n$ has $2^n$ leaf nodes.
Coin Tossing: Analysis

Consider a sequence of $n$ tosses of a fair coin. For example,

\[ HHTHTTTTHH \cdots T \]

This sequence corresponds to a particular leaf node of a binary tree of depth $n$. Since the coin is fair, and there are $2^n$ different leaf nodes on this tree, this sequence of $n$ tosses occurs with probability $1/2^n$.

Let $k$ denote the number of $H$'s in the above sequence. How many $T$'s occur?

The number of sequences of $n$ symbols that have $k$ $H$'s and $(n - k)$ $T$'s equals the number of anagrams of that can be created using these symbols:

\[
\frac{n!}{k! (n - k)!}
\]

Thus, the probability of obtaining $k$ heads and $n - k$ tails in a sequence of $n$ tosses (of a fair coin) is

\[
P(k, n) = \frac{n!}{k! (n - k)!} \left( \frac{1}{2} \right)^n.
\]
Coin Tossing: Analysis

\[ P(k, n) = \frac{n!}{k!(n-k)!} \left( \frac{1}{2} \right)^n \]

The diagram shows the probability distribution \( P(k, n) \) for different values of \( n \): 10, 20, 40, 100, and 200. The x-axis represents the number of heads \( k \), ranging from 0 to 200. The y-axis represents the probability, ranging from 0.00 to 0.25.
Pascal’s Triangle

```
  1
 1 1
 1 2 1
 1 3 3 1
 1 4 6 4 1
 1 5 10 10 5 1
 1 6 15 20 15 6 1
 1 7 21 35 35 21 7 1
 1 8 28 56 70 56 28 8 1
 1 9 36 84 126 126 84 36 9 1
 1 10 45 120 210 252 210 120 45 10 1
```
Binomial Coefficients

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!},
\]

\[
\begin{array}{cccccccccc}
\text{n} & n=0 & 1 & n=1 & n=2 & n=3 & n=4 & n=5 & n=6 & n=7 & n=8 & n=9 & n=10 \\
\hline
\text{k} & \text{k=0} & \text{k=1} & \text{k=2} & \text{k=3} & \text{k=4} & \text{k=5} & \text{k=6} & \text{k=7} & \text{k=8} & \text{k=9} & \text{k=10} \\
\end{array}
\]
The Game of Craps: Simple Rules

At least two players are required. The player who rolls the dice is called the *shooter*. The other player is called the *fader*.

Each player places an equal sum into the pot.

Then, the shooter throws two dice.

1. If the the first roll (*come-out*) is a 7 or 11, then the shooter wins the pot. “A *natural*.”

2. If the first roll is a 2, 3, or 12, then the fader wins. “*Craps*.”

3. If the first roll is a 4, 5, 6, 8, 9, or 10, then that roll becomes the *point*. The shooter continues to roll the dice until either:
   1. the *point* appears (in which case, the shooter wins); or,
   2. a 7 appears (in which case, the fader wins).

In this game, does either player have the advantage? If so, by how much?
Pair of Dice

Before we analyze the game of craps, let's study the probabilities of the different outcomes obtainable by a pair of dice. Assume we have a blue die and a red one. The following table shows that there are 36 different outcomes:

If the dice are fair, each event occurs with probability \( \frac{1}{36} \).
Hexary Tree

These outcomes can also be illustrated using a hexary tree (a tree with degree 6). The first level represents the six different outcomes of the blue die, and the second level, the six different outcomes of the red die.

Each leaf node corresponds to a different outcome.
Probability Theory

The modern theory of probability is based on three concepts:

1. The set of elementary events, $\Omega$.
2. The set of all “measurable” events: all subsets of $\Omega$.
3. A probability measure $P$, which assigns a value between 0 and 1 to each subset.
Digression: What is a set?

Recall that a set is a collection of elements. Like many concepts that we consider, sets are abstract. They can be used to describe many different collections. For example, the following sets are easy to enumerate:

- The set of days of the week is \( \{ \text{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday} \} \).
- The set of outcomes of a single coin toss is \( \{ \text{Heads, Tails} \} \).
- The set of suits in a standard deck of playing cards is \( \{ \spadesuit, \heartsuit, \diamondsuit, \clubsuit \} \).
- The set of odd positive integers less than 10 is \( \{1, 3, 5, 7, 9\} \).

We can also speak of sets that are harder to enumerate, like

- The set of cell phones on planet Earth.
- The set of games that can be played by two people.
- The set of possible configurations of a 3 \( \times \) 3 \( \times \) 3 Rubik’s cube.
- The set of possible games of craps.
- The set of integers \( \mathbb{Z} \), or the set real numbers, \( \mathbb{R} \).
Digression: Membership

We will often want a simple way to state that a particular element is contained by some set, for example, that $13$ is an integer. A common notation is to use the symbol $\in$ to designate membership.

Thus

$$13 \in \mathbb{Z}$$

and if $\Omega$ represents the set of elementary events for the outcome of a pair of dice, then

$$(6, 6) \in \Omega$$

indicates that double-sixes is a possible outcome. Likewise, one can write

$$(7, 0) \notin \Omega$$

to indicate that $(7, 0)$ is *not* an element of the set $\Omega$. 
Subsets

A set $A$ is said to be a subset of another set $B$ if every element of $A$ is also an element of $B$.

We use the notation

$$A \subseteq B \quad \text{or} \quad B \supseteq A$$

to indicate that $A$ is a subset of $B$.

For example,

$$\{1, 3, 5\} \subseteq \{1, 2, 3, 4, 5, 6\},$$

and

$$\{1, 2, 3\} \subseteq \{1, 2, 3\}.$$
Digression: Cardinality

The *cardinality* of a set is the number of members it contains. Thus,

- The cardinality of the set of elementary events for a single coin toss, i.e., \{Heads, Tails\} equals 2.
- The cardinality of the set of configurations of a standard $3 \times 3 \times 3$ Rubik’s cube is $43,252,003,274,489,856,000$.
- The cardinality of the set of elementary events for a pair of labeled dice equals 36.
- The cardinality of the set of elementary events for a sequence of 10 coin tosses equals $2^{10} = 1024$.

Absolute value notation $|\cdots|$ is often used to denote cardinality.

Thus, if $\Omega = \{\text{Heads, Tails}\}$ then $|\Omega| = 2$. 
Elementary Events

A mathematician would call each possible roll, or outcome, an elementary event. The set of elementary events, \( \Omega \) (pronounced “omega”), is a set that contains every possible elementary event. (For any given outcome, one (and only one) element of \( \Omega \) occurs.

Thus, for a pair of dice, \( \Omega \) has 36 elements:

\[
\Omega = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), \\
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), \\
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), \\
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), \\
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}
\]
A probability distribution, \( P \), assigns a probability to each elementary event, such that:

- Each probability is greater than or equal to zero.
- All of the probabilities add up to 1.

In our example, these probabilities all equal \( \frac{1}{36} \):

\[
P\{(1, 1)\} = \frac{1}{36}, \quad P\{(1, 2)\} = \frac{1}{36}, \quad \ldots, \quad P\{(6, 6)\} = \frac{1}{36}.
\]
Events and Probabilities

Sometime we want to consider other events that aren’t explicitly contained in \( \Omega \). For example, the event that the numbers shown on the two dice add up to 7. A *measurable event*, or *event* (for short), is any *subset* of \( \Omega \).

For example, let’s let \( S_7 \) denote the event that the sum of the two dice equals 7. Then,

\[
S_7 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}
\]

We often use the notation \( \subset \) to indicate that one set is a subset of another. Thus \( S_7 \subset \Omega \) denotes that the set \( S_7 \) is a subset of set \( \Omega \), i.e. that every element of \( S_7 \) is also an element of \( \Omega \).

The *probability of an event* is obtained by computing the sum of the probabilities of its elements. Thus,

\[
P(S_7) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{1}{6}.
\]
Events and Probabilities (cont.)

As we assume the dice are fair, each elementary event has the same probability of $\frac{1}{36}$.

Thus, we can compute the probability of rolling a sum of seven, as

$$P(S_7) = |S_7| \times \frac{1}{36} = 6 \times \frac{1}{36} = \frac{1}{6}.$$ 

Let $B_1$ denote the event that the blue die shows a one, i.e.,

$$B_1 = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\}.$$ 

and let $R_6$ denote the probability that the red die shows a six:

$$R_6 = \{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)\}.$$ 

Likewise if $D$ denotes the outcome “doubles”, i.e.,

$$D = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}.$$ 

Note that,

$$P(D) = |D| \times \frac{1}{36} = \frac{1}{6}.$$
Two Special Events

There are two special events:

- The *null event*, $\emptyset$, also known as the *empty set*.
- The *universal event*, $\Omega$, i.e., the entire set of elementary events.

It is always true that,

\[ P(\emptyset) = 0 \text{ and } P(\Omega) = 1. \]

In other words, *something always happens.*
The Algebra of Events

As events are described by subsets, it is possible to obtain new events by applying set operations. Let, \( A \) and \( B \) denote two events (or subsets of \( \Omega \)). The three most basic set operations are

- **union**: \( A \cup B \) = the subset of elementary events that occur in either \( A \) or \( B \).
- **intersection**: \( A \cap B \) = the subset of elementary events that occur in both \( A \) and \( B \).
- **complement**: \( \overline{A} \) = represents the subset of elementary events that are not in \( A \).

Set notation is often used to construct events from combinations of other events. For example, let \( D \) denote the event that a pair of dice lands “doubles.” That is

\[
D = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}.
\]

Since \( D \) is a subset of \( \Omega \) (that is \( D \subset \Omega \)) we can evaluate the probability that \( D \) occurs by adding up the probabilities of its elementary events:

\[
P(D) = P\{(1, 1)\} + P\{(2, 2)\} + P\{(3, 3)\} + P\{(4, 4)\} + P\{(5, 5)\} + P\{(6, 6)\} = \frac{1}{6}.
\]
Independence

Another way to approach the pair-of-dice problem, is to look at each die separately:
For the blue die,

\[ P\{B_1\} = P\{(1, 1)\} + P\{(1, 2)\} + P\{(1, 3)\} + P\{(1, 4)\} + P\{(1, 5)\} + P\{(1, 6)\} = \frac{1}{6}. \]

Likewise, for the red die,

\[ P\{R_6\} = P\{(1, 6)\} + P\{(2, 6)\} + P\{(3, 6)\} + P\{(4, 6)\} + P\{(5, 6)\} + P\{(6, 6)\} = \frac{1}{6}. \]

Note that,

\[ P(B_1 \cap R_6) = P\{(1, 6)\} = \frac{1}{36}, \quad \text{and,} \quad P(B_1) \cdot P(R_6) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}. \]

Two events \( A \) and \( B \) are said to be independent if

\[ P(A \cap B) = P(A) \cdot P(B). \]

Events that are not causally related are usually independent.
Back to Craps

The shooter wins if one of the following eight events occurs:

\[ E_4 = \{ \text{first roll} = 4 \text{ and the next 4 occurs before a 7} \} \]
\[ E_5 = \{ \text{first roll} = 5 \text{ and the next 5 occurs before a 7} \} \]
\[ E_6 = \{ \text{first roll} = 6 \text{ and the next 6 occurs before a 7} \} \]
\[ E_7 = \{ \text{first roll} = 7 \} \]
\[ E_8 = \{ \text{first roll} = 8 \text{ and the next 8 occurs before a 7} \} \]
\[ E_9 = \{ \text{first roll} = 9 \text{ and the next 9 occurs before a 7} \} \]
\[ E_{10} = \{ \text{first roll} = 10 \text{ and the next 10 occurs before a 7} \} \]
\[ E_{11} = \{ \text{first roll} = 11 \} \]

Since it is impossible for more than one of these to occur at the same time,

\[ P\{\text{shooter wins}\} = P(E_4) + P(E_5) + \cdots + P(E_{11}). \]
Events $E_7$ and $E_{11}$ correspond to a *natural*. Since these sets are finite, the probability of each event is easy to evaluate:

Since there are six ways to roll a 7,

$$P\{E_7\} = \frac{6}{36} = \frac{1}{6}. $$

Similarly, since there are only two ways to roll an 11,

$$P\{E_{11}\} = \frac{2}{36} = \frac{1}{18}. $$

Thus the probability of rolling a *natural* is

$$P\{\text{natural}\} = P\{E_7\} + P\{E_{11}\} = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}. $$
Evaluating Harder Probabilities

Because each of the events $E_4, E_5, E_6, E_8, E_9,$ and $E_{10}$ are infinite, it is a little harder to evaluate their probabilities. It is useful to know how to sum a geometric series

$$s = 1 + x + x^2 + x^3 + x^4 + \cdots.$$ 

A simple example of this occurs when $x = 1/2$:

$$s = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^k} + \cdots.$$ 

This case can be evaluated geometrically.
Geometric Series

We will evaluate a general geometric series

\[ s = 1 + x + x^2 + x^3 + x^4 + \cdots. \]

using the *principle of self similarity*. Multiplying both sides by \( x \) and then adding 1, yields

\[
1 + x \cdot s = 1 + x \cdot (1 + x + x^2 + x^3 + x^4 + \cdots) \\
= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = s.
\]

Solving the equation

\[ 1 + x \cdot s = s, \]

yields \[ 1 = s - x \cdot s = (1 - x)s, \]
or

\[ s = \frac{1}{1 - x}. \]

Thus,

\[ 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1 - x}. \]
Evaluating Harder Probabilities (cont.)

As an example, note that

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots = \frac{1}{1 - \frac{1}{2}} = 2,$$

and,

$$1 + 0 + 0^2 + 0^3 + 0^4 + \cdots = \frac{1}{1 - 0} = 1.$$
Analysis of Craps (cont.)

Now we consider event $E_4$. Let,

\[
p_4 = \text{probability of rolling a four with two dice}
\]

\[
= P\{(1, 3)\} + P\{(2, 2)\} + P\{(3, 1)\} = \frac{3}{36},
\]

\[
q = \text{probability of rolling a seven} = \frac{1}{6},
\]

\[
r = \text{probability of rolling anything but a four or seven} = 1 - p_4 - q.
\]

Since consecutive rolls define independent events,

\[
P(E_4) = P\{\text{first roll } 4 \text{ and the next 4 occurs before a 7}\}
\]

\[
= P\{\text{first roll } 4\} \cdot P\{\text{next 4 occurs before the first 7}\}
\]

The first factor is just

\[
P\{\text{first roll } 4\} = p_4.
\]
Analysis of Craps (cont.)

The second factor is

\[ P\{4 \text{ occurs before } 7\} = P\{\text{next roll } = 4\} \]
\[ + P\{\text{next roll is neither } 4 \text{ nor } 7, \text{ and following roll } = 4\} \]
\[ + P\{\text{next two rolls are neither } 4 \text{ nor } 7, \text{ and following roll } = 4\} \]
\[ + P\{\text{next three rolls are neither } 4 \text{ nor } 7, \text{ and following roll } = 4\} \]
\[ + \cdots + \]
\[ = p_4 + rp_4 + r^2 p_4 + r^3 p_4 + \cdots \]
\[ = p_4 \left(1 + r + r^2 + r^3 + \cdots\right) \quad \text{A geometric series!} \]
\[ = p_4 \frac{1}{1 - r} \quad \text{(Now recall that } r = 1 - p_4 - q.) \]
\[ = \frac{p_4}{p_4 + q} = \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{6}} = \frac{1}{3} \]
A Heuristic Argument

In a certain way, it is quite reasonable that

$$P\{4 \text{ occurs before a } 7\} = \frac{p_4}{p_4 + q} = \frac{1}{3}.$$  

Consider the sequences of rolls that end with the first occurrence of 4 or 7. For every sequence that ends with its first 4, e.g.,

$$2, 8, 3, 9, 5, 5, 9, 4,$$

there exists a corresponding sequence that ends with its first 7,

$$2, 8, 3, 9, 5, 5, 9, 7.$$

However, there are twice as many ways to roll 7, than 4. Thus the probability of the first sequence (ending in 4) is half as likely as the second (ending in 7). Thus

$$P\{4 \text{ occurs before a } 7\} = \frac{1}{2} \cdot P\{7 \text{ occurs before a } 4\} = \frac{1}{2} \left( 1 - P\{4 \text{ occurs before a } 7\} \right)$$

Thus,

$$P\{4 \text{ occurs before a } 7\} = \frac{1}{3}.$$
Probability that the shooter wins

\[ P(E_4) = P(\{ \text{first roll} = 4 \text{ and the next 4 occurs before a 7} \}) \]
\[ = P(\{4\}) \cdot P(\{4\} \text{ before } \{7\}) = \frac{1}{12} \cdot \frac{1}{3} = \frac{1}{36} \]

\[ P(E_5) = P(\{ \text{first roll} = 5 \text{ and the next 5 occurs before a 7} \}) \]
\[ = P(\{5\}) \cdot P(\{5\} \text{ before } \{7\}) = \frac{4}{36} \cdot \frac{4}{10} = \frac{2}{45} \]

\[ P(E_6) = P(\{ \text{first roll} = 6 \text{ and the next 6 occurs before a 7} \}) \]
\[ = P(\{6\}) \cdot P(\{6\} \text{ before } \{7\}) = \frac{5}{36} \cdot \frac{5}{11} = \frac{25}{396} \]

\[ P(E_7) = P(\{ \text{first roll} = 7 \}) = \frac{1}{6} \]

\[ P(E_8) = P(\{ \text{first roll} = 8 \text{ and the next 8 occurs before a 7} \}) = \frac{25}{396} \]

\[ P(E_9) = P(\{ \text{first roll} = 9 \text{ and the next 9 occurs before a 7} \}) = \frac{2}{45} \]

\[ P(E_{10}) = P(\{ \text{first roll} = 10 \text{ and the next 10 occurs before a 7} \}) = \frac{1}{36} \]

\[ P(E_{11}) = P(\{ \text{first roll} = 11 \}) = \frac{2}{36} = \frac{1}{18} \]
Probability that the Shooter Wins

\[ P(\{\text{shooter wins}\}) = P(E_7) + P(E_{11}) + 2(P(E_4) + P(E_5) + P(E_6)) \]

\[ = \frac{1}{6} + \frac{1}{18} + 2 \left( \frac{1}{36} + \frac{2}{45} + \frac{25}{396} \right) \]

\[ = \frac{244}{495} = 0.4929 \]

Similarly,

\[ P(\{\text{house wins}\}) = 1 - P(\{\text{shooter wins}\}) \]

\[ = 1 - \frac{244}{495} = \frac{251}{495} = 0.5071 \]
Expectation

The *Expected Reward*, $E$, is defined as the product of the value of the reward times the probability of winning the reward:

$$E = \text{Reward} \times P(\text{Winning}).$$

If the shooter is playing a game of craps, and the pot contains $100, then the expected reward (for the shooter) would be

$$E_{\text{shooter}} = 100 \times \frac{244}{495} = 49.29.$$

Thus, if the game were fair, it would only cost the shooter $49.29 for the chance to win $100. Craps, like life however, is not fair, as it usually would cost the shooter $50.00 to play for a $100 pot. Note that

$$E_{\text{fader}} = 100 \times \frac{251}{495} = 50.71.$$

Thus, on average, the fader earns a profit of $0.71 per game.
Gambling Odds: $f$ to $s$

Gamblers often represent a probability as an *odds ratio*, expressed as “$f$ to $s$”, where $f$ and $s$ are two positive numbers. This sort of odds ratio is used when all money is exchanged after the outcome of the event has been determined. For example, if Alice accepts a $200 bet from Bob at 3 to 1 odds that the next coin toss will land tails, then Alice wins $600 from Bob if the event $T$ occurs. Otherwise, Alice loses $200 to Bob. (Note that no money is exchanged before the coin toss.)

The expected value of this bet (for Alice) is thus

$$E = 600P(T) - 200P(H).$$

Note that a positive loss corresponds to a negative reward. If the coin turns out to be fair then Alice is said to have the advantage, as $E = 200$.

A bet placed with an odds ratio $f$ to $s$ is said to be fair, if the probabilities of success $p_s$, and of failure, $p_f$, satisfy

$$p_s = \frac{s}{f + s}, \quad \text{and,} \quad p_f = \frac{f}{f + s}.$$

In this case the expected value of the bet is

$$E = R_s p_s + R_f p_f = f \frac{s}{f + s} + (-s) \frac{f}{f + s} = 0.$$
Gambling Odds: \( f \) to \( s \)

For example

- The odds of rolling double-sixes with two dice is 35 to 1.
- The odds that a fair coin lands heads is 1 to 1.
- The odds that a PowerBall Lottery ticket matches the winning numbers is 146,107,961 to 1.
- The odds of being dealt blackjack is 631 to 32.

If the odds of winning are \( f \) to \( s \), then the probability of winning is

\[
\frac{s}{f + s}
\]
Gambling Payouts

A wager is said to be *fair*, if the net payout matches the odds ratio. For example, if the odds are “3 to 1” then the winner in a fair game should receive $3 for every dollar that he or she bet.

Casinos generally offer wagers to their advantage. Thus, in roulette, were the odds are 38 to 1, the payout might only be 30 to 1.

Gamblers actually define two different kinds of odds ratios. A payout
I went on Saturday last to make a visit in the city; and as I passed through Cheapside, I saw crowds of people turning down towards the Bank, and struggling who should first get their money into the new erected lottery. It gave me a great notion of the credit of our present government and administration, to find people press as eagerly to pay money, as they would to receive it; and at the same time a due respect for that body of men who have found out so pleasing an expedient for carrying on the common cause, that they have turned a tax into a diversion. The cheerfulness of spirit, and the hopes of success, which this project has occasioned in this great city, lightens the burden of the war, and puts me in mind of some games which they say were invented by wise men who were lovers of their country, to make their fellow citizens undergo the tediousness and fatigues of a long siege. I think there is a kind of homage due to fortune, (if I may call it so) and that I should be wanting to my self, if I did not lay in my pretences to her favour, and pay my complements to her by recommending a ticket to her disposal.

The Tatler, January 1709.
Some History

- Ceasar Augustus issued a lottery to raise revenue for city repairs.
- *Il Lotto de Firenze*, Florence Italy, in 1530.
- The Virginia Colonization Company raised revenue by lottery in 1612 (King James I).
- The U.S. Continental Congress issued lottery tickets in November 1776, to raise $500,000 for the Revolutionary War.
PowerBall Lottery

- For each play, pick five white numbers from 1 to 55, plus one number from 1 to 42.
- A match of 5 white numbers plus PowerBall wins the Jackpot.
- A match of 5 white numbers w/o PowerBall wins $200,000.
- A match of 4 white numbers plus PowerBall wins $10,000.
- A match of 4 white numbers w/o PowerBall wins $100.
- A match of 3 white numbers plus PowerBall wins $100.
- A match of 3 white numbers w/o PowerBall wins $7.
- A match of 2 white numbers plus PowerBall wins $7.
- A match of 1 white numbers plus PowerBall wins $4.
- A match of 0 white numbers plus PowerBall wins $3.
Combinations

The probability of winning the PowerBall lottery is related to the number of ways of selecting the five white numbers (from 1 to 55) and the PowerBall number (from 1 to 42). This is called a *combination*.

Consider the simpler problem of selecting three (lottery numbers) from a set of five \{1, 2, 3, 4, 5\}. How many different selections are there?

In this case we can easily list them:

```
123  124  125  134  135  145
234  235  245
345
```

Note that there are a total of ten combinations, and each digit appears in exactly six of the above selections. (This also equals the number of different triangles that can be connected between five distinct vertices.)
Counting Combinations

This number of combinations \(10\) turns out to equal the number of distinct anagrams of “AAABB” (i.e., strings with three As and two Bs), which we know how to evaluate as

\[
\frac{5!}{3! \cdot 2!} = 10
\]

This “coincidence” occurs because there is a \textit{one-to-one correspondence} between each combination of three numbers, and each permutation of these five letters:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>↔</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>124</td>
<td>↔</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>125</td>
<td>↔</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>134</td>
<td>↔</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>135</td>
<td>↔</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>145</td>
<td>↔</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>234</td>
<td>↔</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>235</td>
<td>↔</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>245</td>
<td>↔</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>345</td>
<td>↔</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>A</td>
</tr>
</tbody>
</table>
Counting Combinations (cont.)

With a little thought, you should be able to convince yourself that this one-to-one correspondence can be used to count the number of ways of selecting \( k \) objects from a set of size \( n \), where \( k \) and \( n \) are arbitrary numbers, such that \( 0 \leq k \leq n \). Simply count the number of anagrams of \( k \) As and \((n - k)\) Bs:

\[
\frac{n!}{k! (n-k)!}.
\]

This number is so useful that a special notation is used. The symbol \( \binom{n}{k} \) represents the number of ways of selecting \( k \) objects from a set of size \( n \), and for short is called “\( n \) choose \( k \)”. Thus,

\[
\binom{n}{k} = \frac{n!}{k! (n-k)!}.
\]

(This number is also called a binomial coefficient and appears in the \( k \)-th column of the \( n \)-th row in Pascal’s Triangle.)
Back to the PowerBall

Thus the number of ways of selecting 5 white numbers from the set \{1, 2, \ldots, 55\}, is

\[
\binom{55}{5} = \frac{55!}{5! \cdot 50!} = 3478761.
\]

Similarly, the number of ways of selecting 1 PowerBall number from the set \{1, 2, \ldots, 42\}, is

\[
\binom{42}{1} = \frac{42!}{1! \cdot 41!} = 42.
\]

For every way of selecting 5 white numbers, there are thus 42 ways of selecting the PowerBall number. Thus, but the multiplication principle, the number of different lottery outcomes is

\[
\binom{55}{5} \times \binom{42}{1} = 146107962.
\]

For this problem, each outcome is an elementary event. Thus \(|\Omega| = 146107962\).
Winning the Jackpot

In a given PowerBall Lottery, there are 5 winning white numbers and 50 losing white numbers. Similarly, there is 1 winning PowerBall number, and 41 losing PowerBall numbers.

The number of elements in $\Omega$ that match all five winning numbers, and the winning PowerBall number is

$$\binom{5}{5} \times \binom{50}{0} \times \binom{1}{1} \times \binom{41}{0} = 1.$$ 

Since each outcome is equally likely, the probability that a given ticket wins the Jackpot is

$$p_1 = P(\text{Jackpot}) = \frac{1}{146107962} \approx 6.84425 \times 10^{-9}.$$
Winning the Second Largest Prize

What is the probability of winning the second prize of $200,000? 

The number of elementary events that match five of the winning white numbers, but not the PowerBall is

\[
\binom{5}{5} \times \binom{50}{0} \times \binom{1}{0} \times \binom{41}{1} = 41. 
\]

Thus,

\[
p_2 = P(\text{Second Prize}) = \frac{41}{146107962} = 2.80614 \times 10^{-7}. 
\]
Winning the Third Largest Prize

A ticket that matches 4 white numbers and the PowerBall wins a $10,000 prize.

The number of elementary events in $\Omega$ that are members of this event are

$$\binom{5}{4} \times \binom{50}{1} \times \binom{1}{1} \times \binom{41}{0} = 250.$$ 

Thus,

$$p_3 = P(\text{Third Prize}) = \frac{250}{146107962} = 1.71106 \times 10^{-6}.$$
Winning the Fourth Prize

There are two ways to win the fourth prize of $100. Either:

- **A.** the ticket matches 4 white numbers without the PowerBall, or
- **B.** the ticket matches 3 white numbers with the PowerBall.

The number of ways that event $A$ occurs is,

$$\binom{5}{4} \times \binom{50}{1} \times \binom{1}{0} \times \binom{41}{1} = 10250.$$ 

Similarly, the number of ways that event $B$ occurs is,

$$\binom{5}{3} \times \binom{50}{2} \times \binom{1}{1} \times \binom{41}{0} = 12250.$$ 

Thus,

$$p_4 = P(\text{Fourth Prize}) = \frac{10250 + 12250}{146107962} = 1.5400 \times 10^{-4}.$$
Winning the Fifth Prize

There are two ways to win the fifth prize of $7. Either

A. the ticket matches 3 white numbers and not the PowerBall, or

B. it matches 2 white numbers and the PowerBall.

The number of ways that event A occurs is

\[
\binom{5}{3} \times \binom{50}{2} \times \binom{1}{0} \times \binom{41}{1} = 502250.
\]

The number of ways that event B occurs is

\[
\binom{5}{2} \times \binom{50}{3} \times \binom{1}{1} \times \binom{41}{0} = 196000.
\]

Since no elementary event falls into both categories,

\[
p_5 = P(\text{Fifth Prize}) = \frac{502250 + 196000}{146107962} = 0.004779.
\]
Winning the Sixth Prize

A ticket that matches 1 white number and the PowerBall wins a $4 prize. The number of elementary events in $\Omega$ of this type is

$$\binom{5}{1} \times \binom{50}{4} \times \binom{1}{1} \times \binom{41}{0} = 1151500.$$ 

Thus,

$$p_6 = P(\text{Sixth Prize}) = \frac{1151500}{146107962} = 0.00788116.$$
Winning the Seventh Prize

A ticket that matches 0 white numbers and the PowerBall wins a $3 prize. The number of elementary events in $\Omega$ of this type is

$$\binom{5}{0} \times \binom{50}{5} \times \binom{1}{1} \times \binom{41}{0} = 2118760.$$ 

Thus,

$$p_7 = P(\text{Seventh Prize}) = \frac{2118760}{146107962} = 0.0145013.$$
Expectation

Consider a lottery where there are multiple ways to win:

- Reward $R_1$ with probability $p_1$,
- Reward $R_2$ with probability $p_2$,
- \vdots
- Reward $R_n$ with probability $p_n$.

The expected reward is defined as

$$E = R_1 p_1 + R_2 p_2 + \cdots + R_n p_n.$$ 

For the PowerBall, let's assume a $5,000,000$ jackpot, for $R_1$. Then the expected reward for a single PowerBall ticket equals,

$$E = (5 \times 10^6) \times (6.84425 \times 10^{-9}) + (2 \times 10^4) \times (2.90614 \times 10^{-7}) + 10^4 \times (1.71106 \times 10^{-6}) + 10^2 \times (1.5400 \times 10^{-4}) + 7 \times 0.004779 + 4 \times 0.0078811 + 3 \times 0.0145013
= 0.22. \quad (Play \ responsibly!)$$
Instant Wins and Raffles

Many state lotteries also sponsor instant win lotteries. Often, one scratches off a decal to reveal if one has won a prize. Usually, one receives zilch. This is just like a ticket raffle.

If each the probability of purchasing (or drawing) each ticket is equal, then, all of the probabilities in the formula for the expected reward,

\[ E = R_1 p_1 + R_2 p_2 + \cdots + R_n p_n, \]

are equal. Thus,

\[ E = (R_1 + R_2 + \cdots + R_n)/n, \]

just the sum of all prizes divided by the number of tickets.
For Further Reading