9. River-Crossing Puzzles

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Alcuin’s Farmer-Wolf-Goat-Cabbage Puzzle

Propositio de lupo et capra et fasciculo cauli:
Homo quidam debebat ultra flavium transferre lupum, capram, et fasciculum cauli.
Et non potuit aliam navem invenire, nisi quae duos tantum ex ipsis ferre valebat.
Praeceptum itaque ei fuerat ut omnia haec ultra illaesa omnino transferret. Dicat,
qui potest, quomodo eis illaesis transire potuit.

So reads the Latin text written by the eighth-century Catholic monk and educator, Alcuin of York, which stands as problem 18 (of 53) in his Propositiones ad acuendos juvenes, or Propositions to Sharpen the Young. In the year 781, Alcuin was invited by Charlemagne to teach at the palace school in Aachen. Some historians credit Alcuin’s pedagogy for the foundation for the earliest European universities. The above puzzle represents one of Alcuin’s most famous problems. Hadley and Singmaster’s translation follows:

Proposition of a wolf, a goat and a bunch of cabbages:
A man had to take a wolf, a goat and a bunch of cabbages across a river. The only boat he could find could only take two of them at a time. But he had been ordered to transfer all of these to the other side in good condition. How could this be done?
Farmer-Wolf-Goat-Cabbage Puzzle

The following diagram represents the initial state, with the Farmer, Wolf, Goat, and Cabbages, each represented by its first initial (F, W, G, C) on the left bank of the river. Note that the Farmer must pilot the boat in each direction, and can carry at most one of the other items. However, if the farmer abandons the Goat with the Cabbage, or the Wolf with the Goat, something will get eaten. Can you find a sequence of moves that will safely bring the Farmer, Wolf, Goat, and Cabbage to the opposite bank of the river?
State representation

We will define a state of the puzzle by the locations of the Farmer (F), the Wolf (W), the Goat (G), and the Cabbages (C). After each crossing of the boat, each entity must be either on the left bank, or on the right bang. By applying the multiplication principle, we find that the number of states equals

\[ n_F \times n_W \times n_G \times n_C = 2 \times 2 \times 2 \times 2 = 16. \]

Each state can be represented in *nested-list* notation, using two sub-lists: the first sub-list describes the entities on the left bank, while the second denotes those entities on the right bank. Thus the initial state is denoted by

\[
((F \ W \ G \ C) \ ()).
\]

and the goal state, by

\[
(() \ (F \ W \ G \ C)).
\]

Note that the order within each sub-list is not important.
Building the State Transition Graph

Which moves are possible from the initial state $((F \ W \ G \ C) ())$? Either the Farmer crosses the river alone by himself, or with one of his three possessions. Thus there are four possible moves, which we represent by the resulting state:

1. $((W \ G \ C) (F))$.
2. $((G \ C) (F \ W))$.
3. $((W \ C) (F \ G))$.
4. $((W \ G) (F \ C))$.

In the above, safe states appear in blue, and lethal states in red. For example, State 2 is lethal because the Goat can eat the Cabbages.

We can now begin to construct the state transition graph, which when completed will contain one node for each of the sixteen possible states of the puzzle. Its edges will connect each pair of states that are accessible to each other by a single move.

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((FWGC) ())
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((GC) (FW))  ((WGC) (F))  ((WC) (FG))  ((WG) (FC))
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The State Transition Graph

By continuing the expansion process, and carefully accounting for each state and transition, we obtain the following state transition graph.

Note that the puzzle has two solutions of minimal length; each requires seven moves.
Applying Trémaux’s Algorithm

As was the case for our graph of the Hampton Court maze, the paths that represent the two solutions to Alcuin’s puzzle are conspicuous. In any event, it is possible to discover at least one of them by applying Trémaux’s algorithm to the graph, treating lethal states as dead ends, and safe states of degree 3 or higher as junctions. For example, a left-bearing mouse following Trémaux’s rules would label the edges of the state transition graph as follows:
The Missionaries and the Cannibals

Three missionaries (M) and three cannibals (C) must cross a river using a boat that can at most hold two passengers. If ever the number of cannibals exceeds the number of missionaries on either bank, then those missionaries will be consumed.

Here it is advantageous to represent each state by a list of three numbers, which we label \((m \ c \ b)\). Here \(m\), \(c\), and \(b\) denote the number of missionaries, cannibals, and boats on the original bank, respectively. Note that (assuming that nobody gets eaten) the number of missionaries, cannibals, and boats on the destination bank are likewise \(3 - m\), \(3 - c\) and \(1 - b\).

Thus the initial state is \((3 \ 3 \ 1)\) and the goal state, \((0 \ 0 \ 0)\). How many states are there? How many of these are “safe?” Construct a state transition graph and solve the puzzle.
References


