12.2 Annuities: A application of sequences

Annuity - equal payments made at equal periods of time

ordinary annuity - payment made at the end of a period and frequency of payment = frequency of compounding payment period - time between payments

term of annuity - beginning of first period to end of last period

Amt. earned \( \Rightarrow \) compound annually

\[ A = P (1 + r)^t \]

\( A = \) Amount
\( P = \) principal (initial investment \#)
\( r = \) APR \% of interest
\( t = \) time (years)

General Compound Interest formula

\[ A = P \left(1 + \frac{r}{m}\right)^{mt} \]

\( m = \) no. of compounding periods/year

- \( m = 1 \) annually
- \( m = 2 \) semi-annually
- \( m = 4 \) quarterly
- \( m = 12 \) monthly
Ex. You deposit $1500 at the end of each year for the next 6 years at 5% APR compounded annually. How much will be in the account at the end of 6 years?

\[
A = 1500 \times (1 + 0.05)^5 = 1500 \times 1.05^5
\]

\[
B = 1500 \times (1.05)^4
\]

\[
C = 1500 \times (1.05)^3
\]

\[
D = 1500 \times (1.05)^2
\]

\[
E = 1500 \times (1.05)
\]

\[
F = 1500 \times x \times 1.05 \times x \times 1.05 \times x \times 1.05
\]

\[
S_n = 1500 + 1500(1.05) + 1500(1.05)^2 + 1500(1.05)^3
\]

\[
+ 1500(1.05)^4 + 1500(1.05)^5
\]

\[
S_n = \frac{a \times (r^n - 1)}{r - 1}
\]

\[
a = 1500 \quad r = 1.05 \quad n = 6
\]

\[
S_n = \frac{1500 \times (1.05^6 - 1)}{1.05 - 1}
\]

\[
= \frac{10,202.87}{0.05}
\]

\[
= 102,028.70
\]
Generalizing:

Amount $S$, of an annuity of payments of $R$ dollars each, made at the end of each period for $n$ interest periods at a rate of interest $i$ per period

$$S = R \left[ \frac{(1+i)^n - 1}{i} \right]$$

with $R =$ payment $\rightarrow \frac{r}{m} \rightarrow$ APR $n = mt$ years

or

$$S = R \cdot \frac{1 - \left( \frac{i}{m} \right)^n}{\frac{i}{m}}$$

$$S = R \left[ \frac{\left( 1 + \frac{r}{m} \right)^{mt} - 1}{\frac{r}{m}} \right]$$

$r =$ APR $t =$ time (years) $m =$ no. of compounding periods/yr.
Annuites (An annuity is an amount, $S$, accumulated when payments of $R$ dollars each are made at the end of each period for $n$ consecutive interest periods at a rate of $i$ percent interest per period.)

$$S_n = R \left[ \frac{(1+i)^n - 1}{i} \right]$$

or

$$S_n = R \cdot \frac{s_n}{i} \cdot \frac{1}{m}$$

$m =$ no. of compounding/yr,

$n =$ no. of periods

1. Jill is an athlete who feels her playing career will last 5 more years. To prepare for her future, she deposits $1,500 at the end of each year for 5 years into an account paying 2% compounded annually. How much will she have on deposit after 5 years?

\[
R = \$1500 \\
R = 5 \\
i = .02 \\
(i = m = 1) \\
S_n = 1500 \left[ \frac{(1.02)^5 - 1}{.02} \right] = \$7806.06
\]

2. You deposit $500 at the end of the month for 2 years into an account with an APR of 6% compounded monthly. How much will you have in the account at the end of 2 years?

\[
R = \$500 \\
R = 2 (12) = 24 \\
i = \frac{.06}{12} = .005 \\
S_{24} = 500 \left[ \frac{(1.005)^{24} - 1}{.005} \right] = \$12715.98
\]

Sinking Fund (A sinking fund is a fund set up to receive periodic payments. These payments plus the interest on them are designed to produce a given total at some time in the future. For example, a corporation might set up a sinking fund to receive money that will be needed to pay off a loan in the future. To solve a sinking fund problem you use the formula for an ordinary annuity and solve for $R$.

Know $S_n$, Solve for $R$

3. Della’s Deli wants to buy an expensive slicer in 4 years. At that time, the slicer should cost $6000. Della plans to save a specific amount of money at the end of each month into an account with an APR of 2% compounded monthly. How much should each payment be?

\[
R = ? \\
R = 4 (12) = 48 \\
i = \frac{.02}{12} = .0016 \\
S_{48} = 6000 \\
6000 = R \left[ \frac{(1+(\frac{.02}{12}))^{48} - 1}{(\frac{.02}{12})} \right] \\
6000 = R \left[ 49.92895865 \right] \\
R = \$120.17
\]
Present Value of an Annuity (We want to find a lump sum \( P \) that must be deposited today at a rate of \( i \) percent per period to produce the same amount as the annuity \( S \) after \( n \) periods.)

\[
\begin{align*}
P (1+i)^n & = S = R \left[ \frac{(1+i)^n - 1}{i} \right] \\
\text{Solve for} \ P & = \frac{P}{(1+i)^n} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] \quad \text{or} \quad P = R \left( 1 + \frac{1}{(1+i)^n} \right)
\end{align*}
\]

4. Grandma Marita gave each grandchild a choice of an annuity with payments of $1,000 deposited at the end of each year at 8% compounded annually for 5 years OR the equivalent present value. What is the present value?

\[
\begin{align*}
R &= \$1,000 \\
n &= 5 \\
i &= 0.08
\end{align*}
\]

\[
P = 1000 \left[ \frac{1 - (1.08)^{-5}}{0.08} \right] = \$3,992.71
\]

Check

\[
A = 3,992.71 (1.08)^5
\]

\[
= \$5,866.60 \quad \text{Compounded for 5 yrs.}
\]

\[
S_5 = 1000 \left[ \frac{(1.08)^5 - 1}{0.08} \right] = \$5,866.60
\]