Math 20 Chapter 11 Notes
Spring 2011

Part I. Probability Density Functions (pdfs) and Cumulative Distribution Functions (CDFs)

1. The probability density function for the score on a fitness test is given by
   \[ f(x) = \frac{x^3}{5000} (10-x), \quad 0 \leq x \leq 10 \] points

   \[ \int_0^2 \frac{x^3}{5000} - \frac{x^4}{5000} \, dx = \frac{1}{2000} x^4 - \frac{1}{2500} x^5 \bigg|_0^2 = 0 \]

   a. Find the probability that Jill scores 2 or less on the fitness test.
      \[ P(X \leq 2) = .00672 \]

   b. Find the probability that Linda scores 3 or less on the fitness test.
      \[ P(X \leq 3) = .03078 \]

   c. Find the probability Jean scores between 4 and 6 on the fitness test.
      \[ P(4 \leq X \leq 6) = \int_4^6 \frac{x^3}{5000} - \frac{x^4}{5000} \, dx = \frac{1}{2000} x^4 - \frac{1}{2500} x^5 \bigg|_4^6 = 0.24992 \]

   d. Find the probability Nancy scores 7 or higher on the fitness test.
      \[ P(X \geq 7) = 1 - P(X < 7) = 1 - F(7) = .47178 \]

2. Find the Cumulative Distribution Function for pdf given in #1.

   \[ \int_0^x \frac{t^3}{5000} - \frac{t^4}{5000} \, dt = \frac{1}{2000} t^4 - \frac{1}{25000} t^5 \bigg|_0^x = .24992 \]

   a. Find the probability that Jill scores 2 or less on the fitness test.
      \[ P(X \leq 2) = F(2) = .00672 \]

   b. Find the probability that Linda scores 3 or less on the fitness test.
      \[ P(X \leq 3) = F(3) = .03078 \]

   c. Find the probability Jean scores between 4 and 6 on the fitness test.
      \[ P(4 \leq X \leq 6) = F(6) - F(4) = .24992 \]

   d. Find the probability Nancy scores 7 or higher on the fitness test.
      \[ P(X \geq 7) = 1 - P(X < 7) = 1 - F(7) = .47178 \]
3. The probability density function for the height of a mature Nagle Oak tree is given by

\[ f(x) = \frac{1}{9} x - \frac{1}{18}, \quad 2 \leq x \leq 5 \text{ feet} \]

\[ \text{r.v. } x = \text{tree height} \]

CDF: \[ F(x) = \int_{a}^{x} \frac{1}{9} t - \frac{1}{18} \, dt = \frac{1}{18} t^2 - \frac{1}{18} t \bigg|_{2}^{x} \]

\[ = \frac{1}{18} x^2 - \frac{1}{18} (x) - \left( \frac{1}{18} (2)^2 - \frac{1}{18} (2) \right) \]

\[ F(x) = \frac{1}{18} x^2 - \frac{1}{18} x - \frac{2}{18} \]

\[ [2, 5] \]

a. Find the Cumulative Density Function for the height of the tree.

\[ F(x) = \frac{1}{18} x^2 - \frac{1}{18} x - \frac{2}{18} \quad [2, 5] \]

b. Find the probability the tree is 4 feet or less at maturity.

\[ P(x \leq 4) = F(4) = \frac{1}{18} (4)^2 - \frac{1}{18} (4) - \frac{2}{18} = \frac{16}{18} - \frac{4}{18} - \frac{2}{18} = \frac{10}{18} = \frac{5}{9} \]

No integration for prob. \( \rightarrow \) done when finding \( F(x) \), CDF

4. The pdf for the number of feet between birds' nests at the Parpart Wildlife Refuge is given by

\[ f(x) = 2xe^{-x^2}, \quad x \geq 0 \text{ feet} \]

\[ \text{r.v. } x = \text{feet between nests} \]

\[ \int_{2}^{5} 2xe^{-x^2} \, dx \]

\[ u = -x^2 \]

\[ du = -2x \, dx \]

\[ -dx = 2 \, dx \]

\[ = -x \left. \bigg|_{2}^{5} \right. \]

\[ \int_{2}^{5} e^{-u} \, du = -e^{-u} \bigg|_{2}^{5} = -e^{-5} + 1 = \frac{1}{e^5} \]

Part II. Expected Value, Variance, Standard Deviation

\[ \begin{array}{c|c|c|c|c|c|c}
X & 0 & 1 & 2 & 3 & 4 & 5 \\
P(x) & 0.2 & 0.4 & 0.1 & 0.05 & 0.15 & 0.1 \\
\end{array} \]

\[ \Rightarrow \text{Expected Value} = \mathbb{E}(X) = \sum x \cdot P(x) \]

\[ = 1.85 \]

\[ \sigma^2 = \text{Var}(X) = \sum (x - \mu)^2 \cdot P(x) \]

\[ = \text{St. dev.} = \sigma = \sqrt{\text{Var}(X)} \]
2. Continuous Case

Given pdf \( f(x) \) on \([A, B]\), then the Expected Value is \( E(x) = \mu = \int_A^B x f(x) \, dx \).

\[ \text{Integral over entire interval of } x = f(x) \]

Variance is \( V(x) = \sigma^2 = \int_A^B (x-\mu)^2 f(x) \, dx = \int_A^B x^2 f(x) \, dx - [E(x)]^2 \) and

\[ \text{Standard deviation is } \sigma = \sqrt{V(x)} \]

Median = \( m \) such that \( \int_A^m f(x) \, dx = \frac{1}{2} \)

\[ \text{middle} \]

2. The length of a petal for the Virtue Daisy is between 1 and 4 inches, given by the pdf

\[ f(x) = \frac{1}{2\sqrt{x}}, \quad 1 \leq x \leq 4 \text{ inches} \]

\[ \text{average} \]

a. Find the expected length of a petal for the Virtue Daisy.

\[ E(x) = \int_1^4 x \left( \frac{1}{2\sqrt{x}} \right) \, dx = \int_1^4 \frac{1}{2} x^{1/2} \, dx = \frac{1}{2} \left[ \frac{2}{3} x^{3/2} \right]_1^4 = \frac{1}{3} \left( 4^{3/2} - 1^{3/2} \right) = \frac{1}{3} (4) - \frac{1}{3} (1) = 1 \frac{2}{3} \]

b. Find the variance and standard deviation for the length of the petal for the Virtue Daisy.

\[ V(x) = \left( \int_1^4 x^2 \left( \frac{1}{2\sqrt{x}} \right) \, dx \right) - \left( \frac{1}{3} \right)^2 = \frac{32}{3} - \frac{16}{9} = \frac{32}{3} - \frac{16}{9} = \frac{32}{3} - \frac{16}{9} = \frac{3}{2} \approx 1.5 \]

\[ \text{middle} \]

c. Find the probability a petal is longer than 1 standard deviation above the mean.

\[ P(X > 3.20) = \int_{3.2}^4 x^{-1/2} \, dx = \left. \frac{1}{\sqrt{2}} \right|_3.2 = 3.2 \]

\[ \text{middle} \]

d. Find the median petal length.

\[ \text{median} = m \Rightarrow \int_1^m \frac{1}{2} x^{-1/2} \, dx = \frac{5}{2} \]

\[ \sqrt{m} - 1 = \sqrt{m} - 1 \]

\[ m = (1.5) \]

\[ m = (1.5)^2 = 2.25 \]
3. The length of time waiting in a supermarket express lane is a pdf

\[ f(x) = \frac{11}{10(x+1)^2}, \quad 0 \leq x \leq 10 \text{ minutes} \]

Note: this is a really challenging problem! Much harder than I would expect on a test.

a. Find the probability the wait is 4 minutes or less.

\[
\begin{align*}
\int_0^4 \frac{11}{10(x+1)^2} \, dx &= \frac{-11}{10} \left. \frac{1}{(x+1)} \right|_0^4 \\
&= \frac{-11}{10 (4+1)} - \frac{-11}{10 (1+0)} \\
&= -\frac{11}{50} + \frac{11}{10} = \frac{-11 + 55}{50} = \frac{44}{50} = \frac{4.4}{50}
\end{align*}
\]

b. Find the expected wait time.

\[
\begin{align*}
E(x) &= \int_0^{10} x \left( \frac{11}{10} (x+1)^{-2} \right) \, dx \\
&= \int_0^{10} \frac{11}{10} x (x+1)^{-2} \, dx \\
&= \frac{11}{10} \left. \frac{1}{(x+1)} + \frac{11}{10} \ln(x+1) \right|_0^{10} \\
&= -\frac{11}{10} (10! + \frac{11}{10} \ln(11) - (\frac{-11}{10} \ln(1) + \frac{11}{10} \ln(11)) \\
&= \frac{11}{10} (10! - \frac{11}{10} \ln(11)) \\
&= 4.4
\end{align*}
\]

c. Find the probability the wait time is less than or equal to the mean.

\[
P(x \leq 1.64) = \int_0^{1.64} \frac{11}{10 (x+1)^2} \, dx = \frac{-11}{10 (x+1)^2} \bigg|_0^{1.64} \\
= \frac{-11}{10 (2.64)} + \frac{11}{10 (1)} = 0.6833 = \frac{41}{60}
\]

d. Find the variance and standard deviation for the wait time.

\[
\begin{align*}
\text{Var}(x) &= \int_0^{10} x^2 \left( \frac{11}{10} (x+1)^{-2} \right) \, dx - \left( \frac{1.64}{10} \right)^2 \\
&= \int_0^{10} \frac{11}{10} x^2 (x+1)^{-2} \, dx \\
&= \frac{11}{10} \left. \frac{1}{(x+1)} + \frac{22}{10} x \ln(x+1) + \frac{22}{10} \left( -\ln(x+1) \right) \right|_0^{10} \\
&= \frac{11}{10} (10! + \frac{22}{10} \ln(11) - \frac{22}{10} + \frac{22}{10} \ln(11)) \\
&= 6.72 - (1.64)^2 = \frac{4.0304}{10} = 0.40304
\end{align*}
\]

\[
\text{St. Dev} = \sqrt{4.0304} = 2.0071
\]

e. Find the probability the wait time is within 1 standard deviation of the mean.

\[
P(1.64 - 1 < x < 1.64 + 1) = P(0.64 \leq x \leq 2.64) = 12 - \frac{22}{10} \ln(11)
\]

\[
\frac{22}{10} \ln(11) \quad \text{solve w/ IBP you don't know this}
\]