A LEbesgue measurable SET THAT IS NOT BOREL

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1. Outline

(1) Ternary Expansions
(2) The Cantor Set
(3) The Cantor Ternary Function (a.k.a. The Devil’s Staircase Function)
(4) Properties of the Cantor Ternary Function
  • Continuous
  • Monotone
  • Maps \( C \) onto \([0, 1]\)
  • Constant on each interval in complement of Cantor set \( C \)
(5) Brief review of Vitali set
(6) Problem #28, pp. 71 - 72, [Roy]

2. Ternary Expansions

We’re quite comfortable using decimal expansions for real numbers, i.e., writing

\[
x = \sum_{n=-\infty}^{N} d_n 10^n = d_N \cdot 10^N + \cdots + d_1 \cdot 10 + d_0 + \frac{d_{-1}}{10} + \cdots
\]

with \( d_n \in \{0, 1, \ldots, 9\} \). But the choice of 10 as our base is quite arbitrary (mathematically, not evolutionarily). In this construction we will be using ternary expansions, that is, writing

\[
x = \sum_{n=-\infty}^{N} t_n 3^n = t_N \cdot 3^N + \cdots + t_1 \cdot 3 + t_0 + \frac{t_{-1}}{3} + \cdots
\]

with \( a_n \in \{0, 1, 2\} \). For instance,

\[
197.2 = 1 \cdot 10^2 + 9 \cdot 10 + 7 + \frac{2}{10}
= 2 \cdot 3^4 + 1 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3 + 2 + \frac{1}{3} + \frac{2}{3^2} + \frac{1}{3^3} + \cdots
= 21022.121\ldots3.
\]

In defining the Cantor ternary function, we will be using ternary expansions for \( x \in [0, 1] \), which can be expressed as

\[
x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}.
\]

(How can you express 1 in this way?)
3. The Cantor Set

Recall that the Cantor set $C$ can be constructed by starting with the interval $[0, 1]$ and iteratively removing the middle third of the remaining intervals. (Draw picture.) At each stage we are removing intervals of the form $\left( \frac{3k - 2}{3^m}, \frac{3k - 1}{3^m} \right)$ with $k \in \{1, \ldots, 3^m - 1\}$.

It can be shown that the Cantor set is also the set of all numbers in $[0, 1]$ that have ternary expansions with no 1s. (Discuss how at $n^{th}$ stage numbers in left, middle and right thirds have 0, 1, and 2 as the $n^{th}$ digit of their ternary expansions, respectively. Use picture.)

4. The Cantor Ternary Function

We define a function $f : [0, 1] \rightarrow [0, 1]$ as follows. Given $x \in [0, 1]$ with $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, let $N$ be the smallest $n$ such that $a_n = 1$. If no such $n$ exists, let $N = \infty$. Define

\[ b_n = \begin{cases} 
\frac{a_n}{2} & \text{if } n < N \\
1 & \text{if } n = N
\end{cases} \]

Define $f$ by

\[ f(x) = f \left( \sum_{n=1}^{\infty} \frac{a_n}{3^n} \right) = \sum_{n=1}^{N} \frac{b_n}{2^n}. \]

Note that we should check that $f$ is well-defined since numbers of the form $\frac{a}{3^N}$ have two ternary expansions. Observe that $f(x) = \sum_{n=1}^{N} \frac{b_n}{2^n}$ is a binary expansion of a number in $[0, 1]$. (Show Mathematica plot.)

Lemma. $f$ is continuous.

Proof. Fix $\epsilon > 0$ and $c \in [0, 1]$. Idea: make $\delta$ small enough so that the ternary expansions of $x$ and $c$ agree sufficiently far. Choose $N$ such that $2^N > 1/\epsilon$. Let $\delta = \frac{1}{3^{N+1}}$. Given $x$ with $|x - c| < \delta$, then $x$ and $c$ have ternary expansions $(x_n)$ and $(c_n)$ such that $x_n = c_n$ for all $n \leq N$. Let $(y_n)$ and $(d_n)$ be the binary expansions of $f(x)$ and $f(c)$, i.e.,

\[ f(x) = \sum_{n=1}^{\infty} \frac{y_n}{2^n}, \quad f(c) = \sum_{n=1}^{\infty} \frac{d_n}{2^n}. \]

Then $y_n = d_n$ for all $n \leq N$. Then

\[ |f(x) - f(c)| = \left| \sum_{n=1}^{\infty} \frac{y_n}{2^n} - \sum_{n=1}^{\infty} \frac{d_n}{2^n} \right| = \left| \sum_{n=1}^{N} \frac{y_n - d_n}{2^n} + \sum_{n=N+1}^{\infty} \frac{y_n - d_n}{2^n} \right| \leq \frac{1}{2^N} < \epsilon. \]

\[ \square \]
Lemma. $f$ is monotone.

Proof. Idea: If $x < y$, then their ternary expansions $(x_n)$ and $(y_n)$ must differ at some point $N$ and at that point $x_N < y_N$. □

Lemma. $f$ is constant on each interval in $[0, 1] \setminus C$.

Proof. Suppose $x, y \in \left(\frac{3k-2}{3^M}, \frac{3k-1}{3^M}\right)$ with ternary expansions $(x_n)$ and $(y_n)$. Without loss of generality, assume that $M$ is the smallest positive integer such that $x_M = 1 = y_M$. Then $x_n = y_n$ for all $n < M$, so

$$f(x) = \sum_{n=1}^{M-1} \frac{(1/2)x_n}{2^n} + \frac{1}{2^M} = \sum_{n=1}^{M-1} \frac{(1/2)y_n}{2^n} + \frac{1}{2^M} = f(y).$$

□

Lemma. $f$ maps $C$ onto $[0, 1]$.

Proof. Suppose $y \in [0, 1]$ has binary expansion $(y_n)$. For each $n$, let $x_n = 2y_n$. Then $x_n = 0$ or $2$ for all $n$, so $x := \sum_{n=1}^{\infty} \frac{x_n}{3^n} \in C$. Since

$$f(x) = \sum_{n=1}^{\infty} \frac{(1/2)x_n}{2^n} = \sum_{n=1}^{\infty} \frac{y_n}{2^n} = y$$

then $f$ maps $C$ onto $[0, 1]$. □

5. Facts About Nonmeasurable Sets

Recall that we constructed the Vitali set $\mathcal{V}$ by choosing representatives for the equivalence classes of the equivalence relation given by $x \sim y$ if and only if $x - y \in \mathbb{Q}$ (i.e., coset representatives for the quotient group $\mathbb{R}/\mathbb{Q}$). We showed that these representatives could be chosen to all lie in $[0, 1]$, but also noted that they could be chosen to lie in any interval $[0, 1/10^n]$ by choosing a suitable decimal approximation. We proved that $\mathcal{V}$ was nonmeasurable by letting $(q_n)$ be an enumeration of the rational numbers in $[0, 1]$ and defining $\mathcal{V}_n = \mathcal{V} + q_n = \{v + q_n : v \in \mathcal{V}\}$. We will use this construction once again in the following propositions.

Proposition. If $E$ is measurable and $E \subseteq \mathcal{V}$, then $\lambda(E) = 0$.

Proof. As in the construction of $\mathcal{V}$, let $(q_i)$ be an enumeration of the rational numbers in $[-1, 1]$. Letting $E_i = E + q_i$ for each $i$, then $(E_i)$ is a disjoint sequence and $\lambda(E_i) = \lambda(E)$ for all $i$. (This follows by the same reasoning used in the construction of $\mathcal{V}$.) Since $E \subseteq \mathcal{V} \subseteq [0, 1]$, then $\bigcup_{i \in \mathbb{Z}_{>0}} E_i \subseteq [-1, 2]$. Then

$$3 \geq \lambda[-1, 2] \geq \lambda \left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \lambda(E_i) = \sum_{i=1}^{\infty} \lambda(E) = \lambda(E) \sum_{i=1}^{\infty} 1 = \left\{\begin{array}{ll} 0 & \text{if } \lambda(E) = 0 \\ \infty & \text{if } \lambda(E) > 0. \end{array}\right.$$ 

Since $\sum_{i=1}^{\infty} \lambda(E) \leq 3$, then $\lambda(E) = 0$. □
Proposition. If $A \subseteq \mathbb{R}$ with $\lambda^*(A) > 0$, then there exists $E \subseteq A$ with $E$ nonmeasurable.

Proof. Without loss of generality, take $A \subseteq [0,1)$. (Since $\lambda^*(A) > 0$, then it must the case that $\lambda^*(A \cap [n,n+1)) > 0$ for some $n$. Let $B := (A \cap [n,n+1)) - n$. Since Lebesgue measure is translation-invariant, then $\lambda^*(B) > 0$ and $B \subseteq [0,1)$.)

For each $i$, let $E_i = A \cap \mathcal{V}_i$ where $\mathcal{V}_i$ is, as before, the translate of $\mathcal{V}$ by $q_i$. For contradiction, suppose that $E_i$ is measurable for all $i$. Then $E_i \subseteq \mathcal{V}_i = \mathcal{V} + q_i$. Since $\mathcal{V}$ is measurable, then $\lambda(E_i) = \lambda(\mathcal{V}) = 0$ by the previous proposition. Thus $\lambda(E_i) = \lambda(\mathcal{V}) = 0$.

Since $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A \cap \mathcal{V}_i) = A \cap \left( \bigcup_{i=1}^{\infty} \mathcal{V}_i \right) \supseteq A \cap [0,1) = A$ then

$$0 < \lambda^*(A) \leq \lambda^* \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} \lambda^*(E_i) = 0,$$

which is a contradiction. Thus $E_i$ is nonmeasurable for some $i$, and $E_i \subseteq A$. □

6. Constructing A Measurable Non-Borel Set

We follow the construction indicated in Exercise 3.28, pp. 71-72 of [Roy]. Let $f$ be the Cantor ternary function as defined above, and let $g(x) = f(x) + x$.

Lemma. $g : [0,1] \to [0,2]$ is a homeomorphism, i.e., $g$ is a continuous bijection with a continuous inverse.

Proof. • One-to-one: $g$ is increasing
  • Continuous: Since $f$ is continuous, then $g$ is a sum of continuous functions, hence continuous.
  • Onto: Since $g(0) = 0$ and $g(1) = 2$ and $g$ is continuous, then $g$ attains every value between 0 and 2 by the Intermediate Value Theorem.
  • Continuous Inverse: Let $h = g^{-1}$. Suppose $U \subseteq [0,1]$ is open. Then $[0,1] \setminus U$ is closed and bounded, hence compact. Since $g$ is continuous, then $g([0,1] \setminus U)$ is compact. Now $g([0,1] \setminus U) = h^{-1}([0,1] \setminus U) = [0,2] \setminus h^{-1}(U)$, so $[0,2] \setminus h^{-1}(U)$ is compact, hence closed and bounded. Then $h^{-1}(U) \subseteq [0,2]$ is open, hence $h$ is continuous. Therefore $g$ is a homeomorphism.

Lemma. $g(\mathcal{C})$ has measure 1.

Proof. Recall that $f$ is constant on any interval in $[0,1] \setminus \mathcal{C}$. Thus for any interval $(a,b) \subseteq [0,1] \setminus \mathcal{C}$, $\lambda(g(a), g(b)) = g(b) - g(a) = f(b) + b - f(a) - a = b - a$. 4
Let $\{I_{n,k}\}_{k=1}^{2^{n-1}}$ denote the collection of intervals removed at stage $n$ in the construction of $C$. Then

$$\lambda([0, 2] \setminus C) = \lambda(g([0, 1] \setminus C)) = \lambda\left(g \left( \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k} \right) \right) = \lambda\left( \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} g(I_{n,k}) \right)$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \lambda(g(I_{n,k})) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \lambda(I_{n,k}) = 1$$

since the total measure of intervals removed is 1. Since $[0, 2] = g(C) \cup ([0, 2] \setminus g(C))$, then

$$2 = \lambda([0, 2]) = \lambda(g(C)) + \lambda([0, 2] \setminus g(C)) = \lambda(g(C)) + 1,$$

hence $\lambda(g(C)) = 1$. □

Since $\lambda(g(C)) > 0$, then there exists a nonmeasurable $E \subseteq g(C)$. Let $A = g^{-1}(E)$. Since $A \subseteq C$, then $\lambda^*(A) \leq \lambda^*(C) = 0$. Thus $A$ has outer measure zero, hence is measurable, but $g(A) = E$ is nonmeasurable.

Since $g^{-1} = h$ is continuous, hence measurable. We claim that $A$ is not a Borel set. For contradiction, suppose $A$ is Borel. Since $h$ is measurable, then $h^{-1}(A) = g(A) = E$ is measurable, which is a contradiction. Therefore $A$ is not a Borel set.

References