ME 239: Rocket Propulsion

Nozzle Thermodynamics and Isentropic Flow Relations

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Assumptions for this Analysis

1. Steady, one-dimensional flow
   • No motor start/stopping issues to be concerned with
   • No radial flow components (quasi-1D)

2. Adiabatic
   • No shocks in nozzle

3. Frictionless
   • No thermal boundary layer
   • No heat loss through nozzle walls

4. Chemical Equilibrium Established in Combustion Chamber
   • Frozen flow usually assumed as flow characteristic time << chemical reaction characteristic time
   • Local Chemical Equilibrium could be possible

5. Ideal Gas
   • Thermally Perfect Gas
   • Calorically Perfect Gas

6. Axial Exhaust Velocity
Rough Outline of Slides

(1) Energy Equation
(2) Isentropic Flow Relations
(3) Nozzle Mass Flow Rate
(4) Nozzle Discharge Coefficient
(5) Area Ratio Function
(6) Thrust
(7) Specific Impulse
(8) Thrust Coefficient
(1) Energy Equation

Subscripts
0 ≡ total or stagnation
1 ≡ combustion or thrust chamber
2 ≡ nozzle exit plane

- Total enthalpy:

\[ h_0 = \text{constant} \quad\Rightarrow\quad h_0 = h + \frac{v^2}{2} = \text{constant} \]

\[ h_0 = h_1 + \frac{v_1^2}{2} = h_2 + \frac{v_2^2}{2} \]

- Rearranging and solving for nozzle exit velocity, \( v_2 \):

\[ v_2 = \sqrt{2(h_1 - h_2) + v_1^2} \]

- Normally it’s safe to assume stagnated conditions in the combustion chamber: \( v_1 = 0 \)

\[ h_0 = h_1 + \frac{v_1^2}{2} \quad\Rightarrow\quad h_0 = h_1 \quad\Rightarrow\quad v_2 = \sqrt{2(h_0 - h_2)} \]
(1) Energy Equation

For a perfect gas with constant specific heats (not functions of $T$):

- **Equation of state for TPG**
  \[ P = \rho RT \]

- **Internal Energy for CPG**
  \[ h = C_p T \]

Including into the velocity relation:

\[ v_2 = \sqrt{2C_p (T_0 - T_2)} \]
\[ v_2 = \sqrt{2C_p T_0 \left(1 - \frac{T_2}{T_0}\right)} \]

Subscripts
0 ≡ total or stagnation
1 ≡ combustion or thrust chamber
2 ≡ nozzle exit plane
(1) Energy Equation

Subscripts
0 ≡ total or stagnation
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- For a perfect gas

\[ C_p = \frac{kR}{k - 1} \]

Specific heat at constant pressure

\[ k = \frac{C_p}{C_v} \]

Specific heat ratio. Frequently denoted as \( \gamma \) in other texts

- Here:

\[ R = \frac{\mathcal{R}}{\mathcal{M}} = \frac{\text{universal gas constant}}{\text{molecular mass}} \]

\[ \mathcal{R} = 8314.3 \text{ [J/kg·mol·K]} \]

\[ \mathcal{R} = 1544 \text{ [ft·lbf/lbm·mol·R]} \]
(1) Energy Equation

Subscripts
0 ≡ total or stagnation
1 ≡ combustion or thrust chamber
2 ≡ nozzle exit plane

- From general *isentropic* (no shocks!) flow relations (see any compressible flow texts for derivation)

\[
\frac{T_1}{T_2} = \left( \frac{P_1}{P_2} \right)^{\frac{k-1}{k}} \quad \text{or} \quad \frac{T_2}{T_0} = \left( \frac{P_2}{P_0} \right)^{\frac{k-1}{k}}
\]

\[
v_2 = \sqrt{2C_pT_0 \left( 1 - \frac{T_2}{T_0} \right)} \quad \text{Eq. 3-16 in text}
\]

Exit velocity for an isentropic nozzle
(1) Energy Equation

Subscripts
0 ≡ total or stagnation
1 ≡ combustion or thrust chamber
2 ≡ nozzle exit plane

- What would be the maximum theoretical exit jet velocity?
- This occurs at an infinite expansion where: \( \frac{P_2}{P_0} = 0 \)
- Thus:

\[
(v_2)_{max} = \sqrt{2 \left( \frac{kR}{k - 1} \right) T_0}
\]
Exhaust velocity \( (v_2) \) and specific impulse \( (I_s) \) are influenced by the ratio of chamber temperature to the molecular mass of propellant species \( (T_1/\mathcal{M}) \).

This ratio plays an important role in optimizing mixture ratios in chemical rockets.

- Increase in energy release can drive this down
- Rich hydrogen content can drive this down

\[
v_2 = \sqrt{\frac{2k\theta_0}{k-1} R' \frac{\theta_1}{\mathcal{M}}} \left[ 1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} \right]
\]

\[
I_s = \frac{v_2}{\theta_0}
\]
(2) Isentropic Flow Relations

Subscripts
0 ≡ total or stagnation
1 ≡ combustion or thrust chamber
2 ≡ nozzle exit plane

- Revisiting the energy equation: \( h_0 = h + \frac{v^2}{2} \)
- Assuming CPG \( h = C_p T \): \( C_p T_0 = C_p T + \frac{v^2}{2} \)

\[
T_0 = T + \frac{v^2}{2C_p}
\]

\[
\frac{T_0}{T} = 1 + \frac{v^2}{2C_p T} \quad \text{Recall that the sound speed and Mach number definitions are:} \quad a^2 = kRT \\
M = \frac{v}{a}
\]

\[
\frac{T_0}{T} = 1 + \frac{k - 1}{2} \frac{v^2}{a^2} \quad \text{Eq. 3-12 in text}
\]
(2) Isentropic Flow Relations

Subscripts
0 ≡ total or stagnation
1 ≡ combustion or thrust chamber
2 ≡ nozzle exit plane

\[ \frac{T_0}{T} = 1 + \frac{k - 1}{2} M^2 \]

Eq. 3-12 in text

Stagnation (aka total) temperature remains constant for adiabatic process

\[ \frac{P_0}{P} = \left( \frac{T_0}{T} \right)^{\frac{k}{k-1}} \]

\[ \frac{P_0}{P} = \left( 1 + \frac{k - 1}{2} M^2 \right)^{\frac{k}{k-1}} \]

Eq. 3-13 in text

Stagnation (aka total) pressure remains constant for isentropic process

\[ \frac{\rho_0}{\rho} = \left( \frac{T_0}{T} \right)^{\frac{1}{k-1}} \]

\[ \frac{\rho_0}{\rho} = \left( 1 + \frac{k - 1}{2} M^2 \right)^{\frac{1}{k-1}} \]

Not in text
(2) Isentropic Flow Relations

Subsonic, Sonic and Supersonic Nozzles

Table 3.1: Nozzle types

<table>
<thead>
<tr>
<th></th>
<th>Subsonic</th>
<th>Sonic</th>
<th>Supersonic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Throat velocity</td>
<td>$v_t &lt; a_t$</td>
<td>$v_t = a_t$</td>
<td>$v_t = a_t$</td>
</tr>
<tr>
<td>Exit velocity</td>
<td>$v_2 &lt; a_2$</td>
<td>$v_2 = v_t$</td>
<td>$v_2 &gt; v_t$</td>
</tr>
<tr>
<td>Mach number</td>
<td>$M_2 &lt; 1$</td>
<td>$M_2 = M_t = 1.0$</td>
<td>$M_2 &gt; 1$</td>
</tr>
<tr>
<td>Pressure ratio</td>
<td>$p_1/p_2 &lt; \left(\frac{k+1}{2}\right)^{k/(k-1)}$</td>
<td>$p_1/p_2 = \frac{p_1}{p_t} = \left(\frac{k+1}{2}\right)^{k/(k-1)}$</td>
<td>$p_1/p_2 &gt; \left(\frac{k+1}{2}\right)^{k/(k-1)}$</td>
</tr>
</tbody>
</table>

Shape

\[
\frac{p_0}{p} = \left[1 + \frac{1}{2}(k - 1)M^2\right]^{k/(k-1)} \quad \text{Eq. 3-13}
\]

For air ($k = 1.4$) and $M = 1$, $P_0/P = 1.89$
Here we will discuss relations describing the mass flow through an isentropic nozzle through analysis of a De Laval nozzle which is a choked supersonic CD nozzle.

We will also introduce a useful parameter based on mass flow known as the mass flow parameter (MFP).

Assumptions (again):
- Steady Quasi-1D flow
- Isentropic Flow
- Perfect Gas
Nozzle Mass Flow Rate

We know that, owing to mass conservation, that the mass flow rate through the nozzle is constant: $\dot{m} = \rho v A$

$$P = \rho RT$$

$$a^2 = kRT$$

$$M = \frac{v}{a}$$

Multiplying through by $P_0/P_0$ and $\sqrt{T_0/T_0}$ and placing $RT$ term under the radical:

$$\dot{m} = \frac{P_0}{P_0} \rho M \sqrt{\frac{kRT}{R^2T^2}} \sqrt{\frac{T_0}{T_0}} A$$

$$\dot{m} = \frac{P}{P_0} P_0 M \sqrt{\frac{k}{RT_0}} \sqrt{\frac{T_0}{T}} A$$

Subscripts

$t \equiv$ throat plane

$1 \equiv$ nozzle entrance plane

$2 \equiv$ nozzle exit plane
(2) Nozzle Mass Flow Rate

Subscripts

t ≡ throat plane
1 ≡ nozzle entrance plane
2 ≡ nozzle exit plane

- Recall from isentropic relations:
\[ \frac{T_0}{T} = 1 + \frac{k - 1}{2} M^2 \quad -\text{and-} \quad \frac{P_0}{P} = \left(1 + \frac{k - 1}{2} M^2\right)^{k-1} \]

\[ \therefore \quad \dot{m} = \left(1 + \frac{k - 1}{2} M^2\right)^{-\frac{k}{k-1}} P_0 M \sqrt{\frac{k}{RT_0}} \sqrt{1 + \frac{k - 1}{2} M^2 A} \]

- Evaluated at the throat (M = 1 and A = At):

\[ \dot{m} = P_0 k A_t \sqrt{\frac{1}{kRT_0} \left(\frac{2}{k + 1}\right)^{\frac{k+1}{k-1}}} \]

As the mass flow through the nozzle is constant, this simplified relation shows how the mass flow rate can be determined from throat geometry and chamber conditions alone.
(2) Nozzle Mass Flow Rate

Subscripts
\[ t \equiv \text{throat plane} \]
\[ 1 \equiv \text{nozzle entrance plane} \]
\[ 2 \equiv \text{nozzle exit plane} \]

- Going back to our Mach number based mass flow equation:

\[
\dot{m} = \left(1 + \frac{k-1}{2} M^2 \right)^{-\frac{k}{k-1}} P_0 M \sqrt{\frac{k}{R T_0}} \sqrt{1 + \frac{k-1}{2} M^2 A}
\]

- Rearranging:

\[
\frac{\dot{m} \sqrt{T_0}}{A P_0} = M \sqrt{\frac{k}{R} \left(1 + \frac{k-1}{2} M^2 \right)^{-\frac{(k+1)}{(k-1)}}}
\]

This is known as the mass flow parameter:

\[
\text{MFP} = \frac{\dot{m} \sqrt{T_0}}{A P_0}
\]

- Evaluated at the sonic throat yields:

\[
\frac{\dot{m} \sqrt{T_0}}{A^* P_0} = \sqrt{\frac{k}{R} \left(\frac{k+1}{2} \right)^{-\frac{(k+1)}{(k-1)}}}
\]
(2) Nozzle Mass Flow Rate

\[
\text{MFP} = \frac{\dot{m}\sqrt{T_0}}{AP_0} = M \sqrt{\frac{k}{R} \left( 1 + \frac{k - 1}{2} M^2 \right)^{-\frac{(k+1)}{(k-1)}}}
\]

\[
\text{MFP}_{\text{max}} = \frac{\dot{m}\sqrt{T_0}}{A^*P_0} = \sqrt{\frac{k}{R} \left( \frac{k + 1}{2} \right)^{-\frac{(k+1)}{(k-1)}}}
\]

\[
\text{MFP}_{\text{max}} \text{ occurs at } M = 1
\]

- \(k = 1.4\)
  \(R = 287 \text{ J/kg-K}\)

- \(k = 1.2\)
  \(R = 378 \text{ J/kg-K}\)
(3) Nozzle Discharge Coefficient

Not yet covered in class
Consider the De Laval nozzle above with fixed $\dot{m}, T_0$, and $P_0$ (isentropic, mass-conserved system).

Let’s analyze the mass flow parameter at locations 1 and 2:

\[
\left( \frac{\dot{m}\sqrt{T_0}}{AP_0} \right)_1 = M_1 \sqrt{\frac{k}{R} \left( 1 + \frac{k - 1}{2} M_1^2 \right)^{\frac{(k+1)}{(k-1)}}}
\]

and

\[
\left( \frac{\dot{m}\sqrt{T_0}}{AP_0} \right)_2 = M_2 \sqrt{\frac{k}{R} \left( 1 + \frac{k - 1}{2} M_2^2 \right)^{\frac{(k+1)}{(k-1)}}}
\]

And then take their ratio:

\[
\frac{\left( \frac{\dot{m}\sqrt{T_0}}{AP_0} \right)_1}{\left( \frac{\dot{m}\sqrt{T_0}}{AP_0} \right)_2} = \frac{M_1}{M_2} \sqrt{\frac{k}{R} \left( 1 + \frac{k - 1}{2} M_1^2 \right)^{\frac{(k+1)}{(k-1)}}} \quad \frac{k}{R} \left( 1 + \frac{k - 1}{2} M_2^2 \right)^{\frac{(k+1)}{(k-1)}}
\]

Note that the minus sign is removed from the exponent.
(4) Area Ratio Function

\[ \frac{A_2}{A_1} = \frac{M_1}{M_2} \sqrt{\frac{1 + \frac{k-1}{2} M_2^2}{1 + \frac{k-1}{2} M_1^2}} \]

- Subscripts
  - \( t \) \( \equiv \) throat plane
  - \( 1 \) \( \equiv \) nozzle entrance plane
  - \( 2 \) \( \equiv \) nozzle exit plane

- Rearranging this relation and maintaining the assumption that \( \dot{m}, T_0, \) and \( P_0 \) are constant:

\[ \frac{A}{A^*} = \frac{1}{M} \sqrt{\frac{2}{k+1} \left(1 + \frac{k-1}{2} M^2\right)^{(k+1)/(k-1)}} \]

- Let's define all \((A)\) \(2\) terms as any point in the nozzle and \((A)\) \(1\) as a sonic throat point \((M_1 = 1 \text{ and } A_1 = A^*)\)

At times we may use the term \( \epsilon \) which represents the nozzle expansion ratio:

\[ \epsilon \equiv \frac{A}{A^*} = \frac{A}{A_t} \]

Recall that \( A_t = A^* \) if nozzle is choked
(4) Area Ratio Function

\[
\frac{A}{A^*} = \frac{A}{A_t} = \frac{1}{M} \sqrt{\frac{2}{k + 1} \left(1 + \frac{k - 1}{2} M^2 \right)^{\frac{k+1}{k-1}}}
\]

For \( k = 1.4 \)
(4) Area Ratio Function

\[
\frac{A}{A^*} = \frac{A}{A_t} = \frac{1}{M} \sqrt{\frac{2}{k+1} \left( 1 + \frac{k-1}{2} M^2 \right)^{\frac{k+1}{k-1}}}
\]

- To solve for \( M \) within the \( A/A^* \) relation one can use a numerical approach to the roots of the function for \( M \)
- Values of \( M \) for various \( A/A^* \) are also readily available and tabulated
- For air \((k=1.4)\):
  - “Modern Compressible Flow,” Anderson
(4) Area Ratio Function

\[
\frac{T_0}{T} = 1 + \frac{1}{2} (k - 1) M^2 \quad \text{Eq. 3-12}
\]

\[
\frac{p_0}{p} = \left[1 + \frac{1}{2} (k - 1) M^2\right]^{k/(k-1)} \quad \text{Eq. 3-13}
\]

- Pressure ratio depends little on \( k \) throughout
- Temperature ratio is influenced much more by variations in \( k \)

**Fig 3-1**: Relationship of area ratio, pressure ratio, and temperature ratio as a function of Mach number through a De Laval nozzle
\[
\frac{A}{A_t} = \frac{1}{M^2} \sqrt{1 + \frac{k - 1}{2} M^2 \frac{k + 1}{k + 1/(k-1)}}
\]

Eq. 3-14 for \(M_1 = 1\)

- Contraction ratio is quite small
- In convergent region there is very little influence from \(k\)
- In supersonic section area ratio becomes large very quickly
- Divergent section significantly influenced by \(k\)

**Fig 3-1:** Relationship of area ratio, pressure ratio, and temperature ratio as a function of Mach number through a De Laval nozzle
Eq. 3-25 and Eq. 3-26

\[
\frac{A_t}{A_y} = \left(\frac{k+1}{2}\right)^{1/(k-1)} \left(\frac{p_y}{p_1}\right)^{1/k} \sqrt{\frac{k+1}{k-1} \left[1 - \left(\frac{p_y}{p_1}\right)^{(k-1)/k}\right]}
\]

\[
\frac{v_t}{v_y} = \sqrt{\frac{k+1}{k-1} \left[1 - \left(\frac{p_y}{p_1}\right)^{(k-1)/k}\right]}
\]
- Solving for area ratio and velocity ratio for a given pressure ratio is straightforward.
- Acquiring pressure ratio from either area ratio or velocity ratio requires numerical solutions as no closed-form solution exists.
- One could also use these charts (helpful in homework problems!)
(5) Thrust

- How do these nozzle flow relations relate to thrust (which is what we’re really interested in)?

- Recall:

  **Thrust Equation**
  
  \[ F = \dot{m}v_2 + A_2(P_2 - P_3) \]

  **Exit Velocity Equation**
  
  \[ v_2 = \sqrt{\frac{2k\mathcal{R}}{k - 1\mathcal{M}}} T_1 \left[ 1 - \left(\frac{P_2}{P_1}\right)^\frac{k-1}{k} \right] \]

  **Mass Flow Equation** (from our MFP relation at the sonic throat):

  \[ \dot{m} = \frac{A^*P_1}{\sqrt{T_1}} \sqrt{\frac{k\mathcal{M}}{\mathcal{R}}} \left(\frac{2}{k + 1}\right)^\frac{(k+1)}{(k-1)} \]

  State 1 is stagnated so 0 = 1
(5) Thrust

• Combining these three relations:

\[ F = \frac{A^* P_1}{\sqrt{T_1}} \left( \frac{kM}{\mathcal{R}} \right) \left( \frac{2}{k+1} \right)^{(k+1)/(k-1)} \left( \frac{2k}{k-1} \frac{\mathcal{R}}{M} T_1 \left[ 1 - \left( \frac{P_2}{P_1} \right)^{k-1} \right] + A_2 (P_2 - P_3) \right) \]

• Bring out the \( A^* P_1 \) term:

\[ F = A^* P_1 \left( \frac{2k^2}{k-1} \left( \frac{2}{k+1} \right)^{(k+1)/(k-1)} \left[ 1 - \left( \frac{P_2}{P_1} \right)^{k-1} \right] + \left( \frac{P_2}{P_1} - \frac{P_3}{P_1} \right) \frac{A_2}{A^*} \right) \]

**Ideal Thrust Equation (Eq. 3-29 in text)**

• This equation applies to ideal rockets of constant \( k \) throughout the expansion process

• This shows that thrust is proportional to the throat area and the chamber pressure (nozzle inlet pressure)

• We also now have a relation for thrust that is independent of chamber temperature and molecular weight of the propellant species
(6) Specific Impulse
The **thrust coefficient** is defined as the chamber pressure, throat area normalized thrust:

\[ C_F = \frac{F}{P_1 A_t} \]

By definition

Introducing our expression for thrust:

\[ F = \dot{m} v_2 + A_2 (P_2 - P_3) \]

\[ C_F = \frac{\dot{m} v_2}{P_1 A_t} + \frac{A_2 (P_2 - P_3)}{P_1 A_t} \]

Recalling an earlier expression for exit velocity derived from the nozzle energy equation:

\[ v_2 = \sqrt{2 C_p T_1 \left(1 - \frac{T_2}{T_1}\right)} \]

\[ \frac{v_2}{\sqrt{T_1}} = \sqrt{2 C_p \left(1 - \frac{T_2}{T_1}\right)} \]

\[ v_2 = \sqrt{2 C_p \left[1 - \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}}\right]} \]

And an expression for the mass flow parameter evaluated at the nozzle throat:

\[ \frac{\dot{m} \sqrt{T_1}}{A_t P_1} = \sqrt{\frac{k}{R} \left(\frac{k + 1}{2}\right)^{\frac{-(k+1)}{(k-1)}}} \]
(7) Thrust Coefficient

- Combining the three highlighted expressions:

\[ C_F = \frac{k}{R} \left( \frac{k + 1}{2} \right)^{-(k+1)/(k-1)} \times 2C_p \left[ 1 - \left( \frac{P_2}{P_1} \right)^{k-1/k} \right] + \epsilon \left( \frac{P_2}{P_1} - \frac{P_3}{P_1} \right) \]

- And substituting the following relation for the specific heat value:

\[ C_p = \frac{kR}{k - 1} \]

\[ C_F = k \sqrt{\frac{2}{k - 1} \left( \frac{2}{k + 1} \right)^{(k+1)/(k-1)}} \left[ 1 - \left( \frac{P_2}{P_1} \right)^{k-1/k} \right] + \epsilon \left( \frac{P_2}{P_1} - \frac{P_3}{P_1} \right) \]

- The thrust coefficient is dimensionless
- A key parameter for analysis as it’s dependent on a gas property, \( k \), the nozzle geometry, \( \epsilon \), and the pressure distribution through nozzle, \( P_1/P_2 \)
- Optimum Thrust Coefficient: peak \( C_F \) for a given motor (\( k \), \( \epsilon \), and \( P_1/P_2 \)) over a constant \( P_1/P_3 \) curve that corresponds to \( P_2 = P_3 \)

Motor thrust can be simply obtained from: \[ F = P_1A_t C_F \]
(7) Thrust Coefficient

Typical variation:

\[ C_F = k \sqrt{\frac{2}{k-1} \left( \frac{2}{k+1} \right)^{(k+1)/(k-1)}} \left[ 1 - \left( \frac{P_2}{P_1} \right)^{k-1} \right] + \epsilon \left( \frac{P_2}{P_1} - \frac{P_3}{P_1} \right) \]

Increasing altitude does increase thrust but the motor is not optimized for that ambient pressure.
(7) Thrust Coefficient

\[ C_F = \frac{k^2}{k - 1} \left( \frac{2}{k + 1} \right)^{(k+1)/(k-1)} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{(k-1)/k} \right] + \frac{p_2 - p_3 A_2}{p_1 A_t} \]

Eq. 3-30
(7) Thrust Coefficient

Eq. 3-30

$k = 1.20$

Line of optimum thrust coefficient
$p_2 = p_3$

Region of flow separation for conical and bell shaped nozzles
Eq. 3-30

(7) Thrust Coefficient
• A nozzle of varying area ratio to maintain a nozzle geometry that will keep the thrust coefficient on the loci of maximum optimal thrust coefficient is ideal

• This is terribly impractical and staging of rocket motors has been adopted as the standard design practice

• This is not to sat that there is still interest in reducing/eliminating the staging process

• Single stage to orbit (SSTO) concepts are still receiving research funding and attention... some of these concepts will be addressed later in the semester