ME 012 Engineering Dynamics

Lecture 17
Angular Momentum, Relation Between Moment of a Force and Angular Momentum, and Principle of Angular Impulse and Momentum
(Chapter 15, Section 5, 6, and 7)

Thursday,
Mar. 21, 2013
Chapter 14: Kinetics of a Particle: Work and Energy
- 14.1 The Work of a Force
- 14.2 Principle of Work and Energy
- 14.3 Principle of Work and Energy for a System of Particles
- 14.4 Power and Efficiency
- 14.5 Conservative Forces and Potential Energy
- 14.6 Conservation of Energy

Chapter 15: Kinetics of a Particle: Impulse and Momentum
- 15.1 Principle of Linear Impulse and Momentum
- 15.2 Principle of Linear Impulse and Momentum for a System of Particles
- 15.3 Conservation of Linear Momentum for a System of Particles
- 15.4 Impact
- 15.5 Angular Momentum
- 15.6 Relation Between Moment of a Force and Angular Momentum
- 15.7 Angular Impulse and Moment Principles
In-Class Activities:
• Applications
• Angular Momentum
• Angular Impulse & Momentum Principle
• Conservation of Angular Momentum
• Problem Solving
APPLICATIONS

Planets and most satellites move in elliptical orbits. This motion is caused by gravitational attraction forces. Since these forces act in pairs, the sum of the moments of the forces acting on the system will be zero. This means that angular momentum is conserved.

The passengers on the amusement-park ride experience conservation of angular momentum about the axis of rotation (the z-axis). As shown on the free body diagram, the line of action of the normal force, N, passes through the z-axis and the weight’s line of action is parallel to it. Therefore, the sum of moments of these two forces about the z-axis is zero.
The angular momentum ($H_O$) of a particle about point $O$ is defined as the “moment” of the particle’s linear momentum about $O$.

Common units: SI [kg·m²/s] FPS [slug·ft²/s]

**SCALAR FORMULATION**

If a particle moves along a curve in the $x$-$y$ plane, the magnitude of the angular momentum about point $O$ acts about the $z$-axis as:

$$(H_O)_z = (d)(mv)$$

**VECTOR FORMULATION**

If a particle moves along a space curve the vector cross-product can be used to find the angular momentum about $O$:

$$H_O = r \times mv = \begin{vmatrix} i & j & k \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix}$$
15.5) Angular Momentum

**VECTOR FORMULATION (cont.)**

\[ H_O = r \times m \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ m v_x & m v_y & m v_z \end{vmatrix} \]

In order to evaluate the \( r \times m \mathbf{v} \) cross-product, the Cartesian components should be expressed so that the angular momentum can be determined by evaluating the determinant.

\[ H_O = (r_y m v_z - r_z m v_y) \mathbf{i} + (r_x m v_z - r_z m v_x) \mathbf{j} + (r_x m v_y - r_y m v_x) \mathbf{k} \]

\[ |H_O| = \sqrt{(r_y m v_z - r_z m v_y)^2 + (r_x m v_z - r_z m v_x)^2 + (r_x m v_y - r_y m v_x)^2} \]
15.6) Relation Between Moment of a Force and Angular Momentum

The resultant force acting on the particle equals the time rate of change of the particle’s linear momentum:

\[ \sum \mathbf{F} = m \dot{\mathbf{v}} \]

The **moments of the forces about** point \( O \) \( (\mathbf{M}_O) \) can be found by performing a cross-product with \( \mathbf{r} \) on both sides:

\[ \sum \mathbf{M}_O = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times m \dot{\mathbf{v}} \]

Differentiating angular momentum w.r.t. time:

\[ \dot{\mathbf{H}}_O = \frac{d}{dt} (\mathbf{r} \times m \mathbf{v}) = (\dot{\mathbf{r}} \times m \mathbf{v}) + (\mathbf{r} \times m \dot{\mathbf{v}}) \]

Thus:

\[ \sum \mathbf{M}_O = \dot{\mathbf{H}}_O \]

This states that the **resultant moment about point** \( O \) of all the forces acting on the particle is equal to the **time rate of change of the particle’s angular momentum** about point \( O \).

From our previous definition of linear momentum \( (\mathbf{L} = m \mathbf{v}) \) we can also see that the resulting force acting on the particle is the time rate of change of the particle’s linear momentum:

\[ \sum \mathbf{F} = \dot{\mathbf{L}} \]
15.7) Principle of Angular Impulse and Momentum

Linear impulse: \( \int_{t_1}^{t_2} F dt \)

Angular Impulse: \( \int_{t_1}^{t_2} M_0 dt = \int_{t_1}^{t_2} (r \times F) dt \)

Moment of a force: \( M_0 = r \times F = r \times m \dot{v} \)

Angular Momentum: \( H_0 = r \times m \dot{v} \)

Principle of Angular Impulse and Momentum:

Initial angular momenta + Sum of all angular impulses = Final angular momenta

VECTOR FORMULATION

\[ m\dot{v}_1 + \sum \int_{t_1}^{t_2} F dt = m\dot{v}_2 \]

\[ (H_0)_1 + \sum \int_{t_1}^{t_2} M_0 dt = (H_0)_2 \]

SCALAR FORMULATION (about z-axis)

\[ m(v_x)_1 + \sum \int_{t_1}^{t_2} F_x dt = m(v_x)_2 \]

\[ m(v_y)_1 + \sum \int_{t_1}^{t_2} F_y dt = m(v_y)_2 \]

\[ (H_0)_1 + \sum \int_{t_1}^{t_2} M_0 dt = (H_0)_2 \]
CONSERVATION OF ANGULAR MOMENTUM

When the sum of angular impulses acting on a particle or a system of particles is zero during the time $t_1$ to $t_2$, the angular momentum is conserved. Thus,

$$\sum_{t_1}^{t_2} \mathbf{M}_o dt = 0$$

Thus,

$$\sum (H_o)_1 = \sum (H_o)_2$$

An example of this condition occurs when a particle is subjected only to a central force. In the figure, the force $\mathbf{F}$ is always directed toward point $O$. Thus, the angular impulse of $\mathbf{F}$ about $O$ is always zero, and angular momentum of the particle about $O$ is conserved.
EXAMPLE 1

The two blocks $A$ and $B$ each have a mass $M_0 = 0.4$ kg. The blocks are fixed to the horizontal rods, and their initial velocity is $v' = 2$ m/s in the direction shown. If a couple moment of $M = 0.6$ N·m is applied about shaft $CD$ of the frame, determine the speed of the blocks at time $t = 3$ s. Assume $a = 0.3$ m, the mass of the frame is negligible, and it is free to rotate about $CD$. Neglect the size of the blocks.
EXAMPLE 1: Solution
EXAMPLE 2

The small cylinder \( C \) has mass \( m_C = 10 \) kg and is attached to the end of a rod whose mass may be neglected. If the frame is subjected to a couple \( M = 8t^2 + 5 \), and the cylinder is subjected to force \( F = 60 \) N, which is always directed as shown, determine the speed of the cylinder when \( t = 2 \) s. The cylinder has a speed \( v_0 = 2 \) m/s when \( t = 0 \).
EXAMPLE 3

The ball has weight $W = 5$ lb and is originally rotating in a circle with velocity $v_B$. As shown, the cord $AB$ has a length of $L = 3$ ft and passes through the hole $A$, which is a distance $h = 2$ ft above the plane of motion. If $L/2$ of the cord is pulled through the hole, determine the speed of the ball, $v_C$, when it moves in a circular path at $C$. 