Math 373 — Exam 1 — Due 03.03.16, 5:00pm

Note: You are to work on this exam on your own. There is no time limit (other than that imposed by the due date). You are welcome to use your course notes, homeworks, and the course textbook. Please do not reference any other materials.

Please return this sheet with your exam.

“I have neither given nor received help on this exam. I have taken it in accordance with the above restrictions and the UVM Code of Academic Integrity.”

Signed:

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Note $\lfloor \varphi \rfloor$: You are welcome to come to me for clarification on any problem.

Note $\lfloor e \rfloor$: 10 points per problems.

Note $\lfloor \pi \rfloor$: Good luck.

(1) Consider the recurrence

$$g_n = \begin{cases} 
0, & \text{for } n < 0, \\
1, & \text{for } n = 0, \\
g_{n-1} + 2g_{n-2} + \cdots + ng_0, & \text{for } n > 0.
\end{cases}$$

Express $g_n$ in terms of the Fibonacci numbers.

(2) Let

$$A(x) = \sum_{n} a_n x^n, \quad B(x) = \sum_{n} b_n x^n, \quad C(x) = \sum_{n} c_n x^n.$$

Assume that $a_n = b_n = c_n = 0$ for $n < 0$. Each of the following three parts is independent from the others; i.e., don’t carry assumptions over from one part to another.

(a) If $c_n = \sum_{j+2k\leq n} a_j b_k$, express $C$ in terms of $A$ and $B$.

(b) If $nb_n = \sum_{k=0}^{n} \left( \frac{2^k a_k}{(n-k)!} \right)$, express $A$ in terms of $B$.

(c) If $r$ is a real number and if $a_n = \sum_{k=0}^{n} \binom{r+k}{k} b_{n-k}$, express $A$ in terms of $B$; then use your formula to find coefficients $f_k(r)$ such that $b_n = \sum_{k=0}^{n} f_k(r) a_{n-k}$.
(3) The Mars Curiosity rover has discovered that organic material on Mars has DNA composed of five symbols, denoted by \{a, b, c, d, e\}, instead of the four components in earthling DNA. The four pairs \(cd, ce, ed,\) and \(ee\) never occur consecutively in a string of Martian DNA, but any string without forbidden pairs is possible. (Thus \(bcda\) is forbidden but \(bdcda\) is just dandy.) Show that the number of length-\(n\) strings of Martian DNA is \([x^n]\left[1 + x \right]/\left(1 - 4x - x^2\right)\) which, incidentally, equals \(F_{3n+2}\) (you don’t need to show this last observation).

Hint: Let \(a_n\) denote the number of length-\(n\) strings of Martian DNA that don’t end in \(c\) or \(e\) and \(b_n\) the number that do.

(4) Let \(\hat{G}(x)\) and \(G(x)\) be the exponential and ordinary generating functions, respectively, for some sequence. Given that (for \(n\) a nonnegative integer) \(\int_0^\infty t^n e^{-t} dt = n!\), write
\[
\int_0^\infty \hat{G}(xt)e^{-t} dt
\]
in terms of \(G(x)^1\).

(5) Compute the coefficients of \(x, x^2\) and \(x^3\) for the compositional inverse of the power series \(x + x^3\).

(6) Let \(r(n)\) denote the number of ways to write \(n\) as a product of two relatively prime factors. 
   (a) Show that \(r(n)/n^2\) is a multiplicative function.
   (b) Find an expression for \(\sum_{n \geq 1} r(n)/n^2\).
   (c) If you’re bored, try to express your answer to (b) in terms of the Riemann zeta function.

(7) Let \(P\) and \(Q\) be locally finite posets. Prove that for \((x, y) \leq (x', y')\) in \(P \times Q\),
\[
\mu_{P \times Q}((x, y), (x', y')) = \mu_P(x, x')\mu_Q(y, y').
\]

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1 You actually need to assume the integral exists. However, this is a technicality I don’t want to bother you with on this exam. So ignore this footnote. Thanks!