Math 373 — Spring 2016 — Homework # 3
Due Tuesday, Feb 9

(1) Practice on combinatorial proofs I: Let $B_n$ denote a $1 \times n$ board. Consider tilings of $B_n$ by $1 \times 1$ squares and $1 \times 2$ dominoes.
(a) Prove that the number of such tilings of $B_n$ is the $(n + 1)$-st Fibonacci number $F_{n+1}$ for $n \geq 0$.
(b) Prove combinatorially that for $n \geq 0$, $\sum_{i=0}^{n} F_{i+1} = F_{n+3} - 1$.

(2) Practice on combinatorial proofs II: Give a combinatorial proof of the identity
$$\sum_{k=0}^{n} k \binom{n}{k} = n 2^{n-1}.$$ 

(3) Practice on combinatorial proofs III: Let $x^n = x(x - 1) \cdots (x - n + 1)$ by the $n$-th falling factorial. Give a combinatorial proof of the identity
$$x^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$ 

(4) Prove that addition in $S$ is continuous.

(5) Suppose $F_n, G_n, P, Q \in K[[x]]$ such that $F_n \to P$ and $G_n \to Q$. Prove that $F_nG_n \to PQ$ by showing directly that the coefficients stabilize appropriately.

(6) Carefully justify the following calculation:
$$\prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1} = \prod_{i=1}^{\infty} (1 - x^{2i}) \prod_{j=1}^{\infty} (1 - x^j)^{-1} = \prod_{k=1}^{\infty} (1 + x^k).$$