Lecture Notes 10  Normal matrices and their eigen systems

0 Orthogonal or unitary matrices

In lecture notes 9 we considered bi-orthogonal bases. We go back to orthogonal bases since they are involved in the diagonalization of an important class of matrices - Hermitian ($A^H = A$) or real symmetric ($A^T = A$).

Let $\{q_1, \ldots, q_p\}$ be an orthonormal basis in $\mathbb{R}^p$.

$$\langle q_i, q_j \rangle = \delta_{ij}$$ (2)

Let $Q = [q_1, q_2, \ldots, q_p]$. We observe that

$$Q^TQ = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_p^T \end{bmatrix} \begin{bmatrix} q_1 & q_2 & \cdots & q_p \end{bmatrix} = \begin{bmatrix} q_1^Tq_1 & q_1^Tq_2 & \cdots & q_1^Tq_p \\ q_2^Tq_1 & q_2^Tq_2 & \cdots & q_2^Tq_p \\ \vdots & \vdots & \ddots & \vdots \\ q_p^Tq_1 & q_p^Tq_2 & \cdots & q_p^Tq_p \end{bmatrix} = I$$ (2)

But since $Q$ is a square matrix

(2) $\Rightarrow$ $Q^T = Q^{-1}$ (3)
Def: A real \( Q \times p \) matrix where \( Q^TQ = I \) is called an orthogonal matrix. (Note that \( Q^TQ = QQ^T = I \)).

Similarly, a complex matrix is called unitary if \( Q^HQ = I \).

That is, \( Q^HQ = QQ^H = I \) \hspace{1cm} (4)

Note 1: Real unitary matrices are orthogonal matrices. So:

Note 2: Columns of a unitary matrix form orthonormal bases in \( \mathbb{C}^p \)

Note 3: \( Q \) is unitary iff \( Q^H \) is unitary. It is easy to show that both rows or columns of a unitary matrix form orthonormal sets.

Thm. 1 (Thm. (7.31) (d,e,f,g))

(i) \( (P \circ Q \text{ are unitary}) \Rightarrow (PQ \text{ is unitary}) \)

(ii) \( (Q \text{ is unitary}) \Rightarrow (\text{ all } |\lambda_{ij}| = 1) \)

(converse not true: \( (0,1) \) not unitary)
(iii) Let $Q$ be unitary. Then for any $x, y \in \mathbb{C}^n$

(a) $\langle Qx, Qy \rangle = \langle x, y \rangle$

(b) $\|Qx\|_2 = \|x\|_2$

Proof: See text for proofs of (i), (ii).

(vi) (a) $\langle Qx, Qy \rangle = (Qx)^H Qy = x^H Q^H Q y = x^H y = \langle x, y \rangle$

(b) $\|Qx\|_2 = \|Qx, Qx\|_2 = \sqrt{\langle Qx, Qx \rangle} = \|x\|_2$

Specifically, transforming vectors with unitary matrices does not change (a) inner products or (b) norms.

Thus, it does not change the angle between vectors or the distance (length) of vectors.

Unitary matrices are important since they do not make angles between vectors smaller, i.e., don't make vectors less, i.e., and they do not transform a small error into a larger (or smaller) error.
Examples of orthogonal (i.e. real unitary) transformations are rotations or reflections in $\mathbb{R}^2$ or $\mathbb{R}^3$. They do not change angles or lengths (see Example (7.39)).

**Def. (7.32 Text)** If there exists a unitary matrix $P$ such that

$$P^*AP = B$$

then $B$ is said to be unitarily similar to $A$. If $P$ is real (and hence orthogonal) then $B$ is said to be orthogonally similar to $A$.

Since the above are special cases of general similarity, all previous results for similar matrices hold here as well.

2. **Schaar Form or Decomposition**: Normal matrices.

We have seen the Gauss-reduced form or row-echelon form provided a useful form for the understanding of matrices. Similarly, the following decompositions are useful for studying "eigen systems".
**Theorem 3** (Thm. 8.2 Text) For any $A$, there is a unitary matrix $Q$ such that
\[ Q^H A Q = T \] (5a)

or equivalently,
\[ A = QTQ^H \] (5b)

where $T$ is an upper-$\Delta$ matrix.

**Proof:**
1) $p = 2$ (in $\mathbb{R}^2$)

$A$ has at least one eigenvalue for which the corresponding at least one eigenvector $v$. We can make $\|v\| = 1$. Now select $y$ such that:

(i) $\langle y, v \rangle = 1$

(ii) $\|y\| = 1$

Consider,
\[ AQ = A [v, y] = [Au, Ay] = [\lambda v, Ay] \]

Now consider
\[ Q^H AQ = [v^H, y^H] [\lambda v, Ay] \]
\[
\begin{bmatrix}
\nu^H \lambda \nu & \nu^H A \nu \\
\nu^H A^H \lambda \nu & \nu^H A^H A \nu
\end{bmatrix} =
\begin{bmatrix}
\lambda & x \\
0 & x
\end{bmatrix} \equiv T
\]

(for \(p=2\))

2) \(p=3\)

Again, let \(A \nu = \lambda \nu\), \(\|\nu\| = 1\).

Select some \(u_1 \perp \nu\), \(u_2 \perp \nu\). Also make \(\langle u_i, u_j \rangle = \delta_{i,j}\) \((i,j = 1,2)\).

\[
A Q = A \begin{bmatrix}
\nu \\
u_1 \\
u_2
\end{bmatrix} =
\begin{bmatrix}
A \nu, A u_1, A u_2
\end{bmatrix}
\]

Consider,

\[
\left(\nu^H A Q = \begin{bmatrix}
\nu^H \\
u_1^H \\
u_2^H
\end{bmatrix} \begin{bmatrix}
\lambda & x & x \\
x & 0 & x \\
x & x & 0
\end{bmatrix}
\right.

Now apply to \(B\) the Schur decomposition

for \(p = 2\) (step 1).
Get,
\[
\begin{bmatrix}
\lambda_A & * & * & * \\
0 & \lambda_B & * & * \\
0 & 0 & \ddots & * \\
0 & 0 & \cdots & \lambda_D
\end{bmatrix}
= T \quad \text{as required.}
\]

Similar procedure for \( p > 4 \).

So: We have shown that any matrix (diagonalizable or not) is unitarily similar to a \( \Delta \) matrix.

3. **Normal matrices and their eigenvalue systems.**

**Def:** A normal matrix \( A \) is a square matrix s.t. \( A^H A = AA^H \),

\( \Rightarrow A^T A = AA^T \) if \( A \) real.

**Thm. 4** (Text Thm. 8-6)

\[\text{A is normal } \iff \text{A is unitarily similar to a diagonal matrix: } A = Q \Lambda Q^H\]

Proof: Text p. 329, 330
Comment: The theorem is a "striking result". Note that a p×p normal matrix can be a real symmetric, Hermitian or unitary matrix. A matrix A is unitarily similar \( \overset{\sim}{\rightarrow} \) to a diagonal matrix:

\[
A = Q \Lambda Q^* \quad \text{i.e.} \quad AA^* = A^*A
\]

\( \Lambda \) contains all the eigenvalues of \( A \) (repeated along with their algebraic multiplicities). If \( A \) and its eigenvalues are real, then \( Q \) can be taken to be real or orthogonal.
Note: If \( A \) is normal, then
\[
A Q = Q \Lambda
\]
where \( \Lambda \) are eigenvalues of \( A \)

\[ \Rightarrow \text{eigenbasis of } A \]

\( \Rightarrow \) eigenbasis \( Q \) of \( A \) can be chosen to be orthonormal.

Corollary: Let \( A \) be Hermitian \( (A^H = A) \). Then,
\[
A = Q \Lambda Q^{-1}
\]

where \( Q \) is unitary \( \Rightarrow \Lambda \) is real.

Proof: (i) \( \Rightarrow A = QTQ^H \) upper \( \Lambda \)

\[
A^H = (QTQ^H)^H = QT^HQ^H
\]

Since \( Q \) is unitary a priori invertible \((Q^H = Q^{-1})\)

Then (i) \( \Rightarrow T = T^H \)

\[
\text{with } \Delta \text{ and } \text{herm } \Delta
\]

\[ \Rightarrow T \text{ must be diagonal } \Rightarrow \text{ its diagonals must be real.} \]

\( \Box \)
So, we have proven the following:

(i) Any Hermitian matrix is diagonalizable,
(ii) its eigenvalues are always real,
(iii) its eigenvectors can be chosen to be orthogonal,
(iv) since real symmetric matrices are a subclass of Hermitian matrices, (i) and (iii) also hold for real symmetric matrices.

Note 1: In Thm. 3 of lecture notes 9, we showed that if \( \lambda_i \) is an eigenvalue of \( A \) corresponding to \( \mathbf{v}_i \),
\[ \mathbf{v}_j \] is an eigenvalue of \( A^T \) corresponding to \( \lambda_j \neq \lambda_i \),
the \( \mathbf{v}_j + \mathbf{v}_i \).

For real symmetric matrices, \( A^T = A \Rightarrow \mathbf{v}_j = \mathbf{v}_i \).

\[ \Rightarrow \mathbf{v}_j + \mathbf{v}_i \text{ if } \lambda_i \neq \lambda_j \]

This agrees with Thm. 4 (see (iii) above).
Now if there are repeated eigenvalues, say \( \lambda_1 = \lambda_2 \), then
the argument of the Notes does not work. \( v_1 \) may not be \( 1 \) \( v_2 \). But we can Gram-Schmidt-orthogonalize
the set \( \{v_1, v_2\} \) to make it orthogonal: \( \{ \frac{v_1}{\sqrt{\lambda_1}}, \frac{v_2}{\sqrt{\lambda_2}} \} \).

The new vector \( \frac{v_2}{\sqrt{\lambda_2}} \) is still an eigenvector since it is a linear combination of 2 eigenvectors.

This is in agreement with the restatement (vii) above, that the eigenvectors of a Hermitian matrix can always be made
orthogonal.

Note 2: Thm. 4 or its corollary or Thm. 4 of Class Notes 8
answer the question of when a matrix is diagonalizable
in the most comprehensive way.