Additional Examples of Chapter 3:
Discrete-Time Signals and Systems in the Frequency Domain

Example E3.1: Determine the DTFT \( X(e^{j\omega}) \) of the causal sequence
\[ x[n] = A\alpha^n \cos(\omega_0n + \phi)\mu[n], \]
where \( A, \alpha, \omega_0, \) and \( \phi \) are real.

**Answer:**
\[ x[n] = A\alpha^n \cos(\omega_0n + \phi)\mu[n] = A\alpha^n \left( \frac{e^{j\omega_0n}e^{j\phi} + e^{-j\omega_0n}e^{-j\phi}}{2} \right)\mu[n]. \]
Therefore,
\[ X(e^{j\omega}) = \frac{A}{2} e^{j\phi} \left( \alpha e^{j\omega_0} \right)^n \mu[n] + \frac{A}{2} e^{-j\phi} \left( -\alpha e^{-j\omega_0} \right)^n \mu[n]. \]

Example E3.2: Determine the inverse DTFT \( h[n] \) of
\[ H(e^{j\omega}) = (3 + 2\cos \omega + 4\cos 2\omega)\cos(\omega / 2) e^{-j\omega/2}. \]

**Answer:**
\[ H(e^{j\omega}) = \left[ 3 + 2 \left( \frac{e^{j\omega} + e^{-j\omega}}{2} \right) + 4 \left( \frac{e^{j2\omega} + e^{-j2\omega}}{2} \right) \right] \left( \frac{e^{j\omega/2} + e^{-j\omega/2}}{2} \right) e^{-j\omega/2} \]
\[ = \frac{1}{2} \left( 2e^{j2\omega} + 3e^{j\omega} + 4 + 4e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega} \right) \]
Hence, the inverse of \( H(e^{j\omega}) \) is a length-6 sequence given by \( h[n] = [1 \ 1.5 \ 2 \ 2 \ 1.5 \ 1] \) \(-2 \leq n \leq 3\).

Example E3.3: Let \( X(e^{j\omega}) \) denote the DTFT of a real sequence \( x[n] \). Determine the inverse DTFT \( y[n] \) of \( Y(e^{j\omega}) = X(e^{j3\omega}) \) in terms of \( x[n] \).

**Answer:**
\[ Y(e^{j\omega}) = X(e^{j3\omega}) = X(e^{j\omega})^3 \]
Now, \( X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \).
Hence,
\[ Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = X(e^{j\omega})^3 = \sum_{n=-\infty}^{\infty} x[n](e^{-j\omega n})^3 = \sum_{m=-\infty}^{\infty} x[m/3]e^{-j3\omega n}. \]
Therefore, \( y[n] = \begin{cases} x[n], & n = 0, \pm 3, \pm 6, \ldots \\ 0, & \text{otherwise} \end{cases} \)

Example E3.4: Without computing the DTFT, determine whether the following DTFT has an inverse that is even or an odd sequence:
\[ x[n] = \begin{cases} n^3, & -N \leq n \leq N, \\ 0, & \text{otherwise}. \end{cases} \]

**Answer:** Since \((-n)^3 = -n^3\), \( x[n] \) is an odd sequence with an imaginary-valued DTFT.
Example E3.5: Let $X(e^{j\omega})$ denote the DTFT of a real sequence $x[n]$. Determine the DTFT $Y(e^{j\omega})$ of the sequence $y[n] = x[n] \oplus x[-n]$.

**Answer:** Let $u[n] = x[-n]$, and let $X(e^{j\omega})$ and $U(e^{j\omega})$ denote the DTFTs of $x[n]$ and $u[n]$, respectively. From the convolution property of the DTFT given in Table 3.4, the DTFT of $y[n] = x[n] \oplus u[n]$ is given by $Y(e^{j\omega}) = X(e^{j\omega})U(e^{j\omega})$. From Table 3.4, $U(e^{j\omega}) = X(e^{-j\omega})$. But from Table 3.2, $X(e^{-j\omega}) = X^*(e^{j\omega})$. Hence, $Y(e^{j\omega}) = X(e^{j\omega})X^*(e^{j\omega}) = |X(e^{j\omega})|^2$ which is real-valued function of $\omega$.

Example E3.6: Without computing the inverse DTFT, determine whether the inverse of the DTFT shown in Figure E3.1 is an even or an odd sequence.

![Figure E3.1](image)

**Answer:** $X(e^{j\omega})$ is a real-valued function of $\omega$. Hence, its inverse is an even sequence.

Example E3.7: A sequence $x[n]$ has zero-phase DTFT as shown in Figure E3.2 Sketch the DTFT of the sequence $x[n]e^{-jn/3}$.

![Figure E3.2](image)

**Answer:** From the frequency-shifting property of the DTFT given in Table 3.4, the DTFT of $x[n]e^{-jn/3}$ is given by $X(e^{j(\omega + \pi/3)})$. A sketch of this DTFT is shown below.
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**Example E3.8**: Consider \{x[n]\} = \{3 \ 0 \ 1 \ -2 \ -3 \ 4 \ 1 \ 0 \ -1\}, \ -3 \leq n \leq 5, with a DTFT given by \(X(e^{j\omega})\). Evaluate the following functions of \(X(e^{j\omega})\) without computing the transform itself: (a) \(X(e^{j0})\), (b) \(X(e^{j\pi})\), (c) \(\int_{-\pi}^{\pi} X(e^{j\omega})d\omega\), (d) \(\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega\), and (e) \(\int_{-\pi}^{\pi} \frac{dX(e^{j\omega})}{d\omega}^2 d\omega\).

**Answer**: (a) \(X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 3 + 1 - 2 - 3 + 4 + 1 - 1 = 3\).

(b) \(X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\pi n} = -3 - 1 - 2 + 3 + 4 - 1 + 1 = 1\).

(c) \(\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0] = -4\pi\).

(d) \(\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 82\pi\). (Using Parseval's relation)

(e) \(\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |\ln x[n]|^2 = 378\pi\). (Using Parseval's relation with differentiation-in-frequency property)