Summary of Mechanics

0) The laws of mechanics apply to any collection of material or ‘body.’ This body could be the overall system of study or any part of it. In the equations below, the forces and moments are those that show on a free body diagram. Interacting bodies cause equal and opposite forces and moments on each other.

I) Linear Momentum Balance (LMB)/Force Balance

Equation of Motion

\[ \sum \vec{F}_i = \dot{\vec{L}} \]  (I)

The total force on a body is equal to its rate of change of linear momentum.

Impulse-momentum (integrating in time)

\[ \int_{t_1}^{t_2} \sum \vec{F}_i \, dt = \Delta \vec{L} \]  (Ia)

Net impulse is equal to the change in linear momentum.

Conservation of momentum (if \( \sum \vec{F}_i = 0 \))

\[ \vec{L} = \vec{0} \Rightarrow \Delta \vec{L} = \vec{L}_2 - \vec{L}_1 = \vec{0} \]  (Ib)

When there is no net force the linear momentum does not change.

Statics (if \( \vec{L} \) is negligible)

\[ \sum \vec{F}_i = \vec{0} \]  (Ic)

If the inertial terms are zero the net force on system is zero.

II) Angular Momentum Balance (AMB)/Moment Balance

Equation of motion

\[ \sum \vec{M}_C = \dot{\vec{H}}_C \]  (II)

The sum of moments is equal to the rate of change of angular momentum.

Impulse-momentum (angular) (integrating in time)

\[ \int_{t_1}^{t_2} \sum \vec{M}_C \, dt = \Delta \vec{H}_C \]  (IIa)

The net angular impulse is equal to the change in angular momentum.

Conservation of angular momentum (if \( \sum \vec{M}_C = \vec{0} \))

\[ \dot{\vec{H}}_C = \vec{0} \Rightarrow \Delta \vec{H}_C = \vec{H}_{C2} - \vec{H}_{C1} = \vec{0} \]  (IIb)

If there is no net moment about point C then the angular momentum about point C does not change.

Statics (if \( \vec{H}_C \) is negligible)

\[ \sum \vec{M}_C = \vec{0} \]  (IIc)

If the inertial terms are zero then the total moment on the system is zero.

III) Power Balance (1st law of thermodynamics)

Equation of motion

\[ \dot{Q} + P = \dot{E}_K + \dot{E}_p + \dot{E}_{\text{int}} \]  (III)

Heat flow plus mechanical power into a system is equal to its change in energy (kinetic + potential + internal).

for finite time

\[ \int_{t_1}^{t_2} \dot{Q} \, dt + \int_{t_1}^{t_2} P \, dt = \Delta E \]  (IIIa)

The net energy flow going in is equal to the net change in energy.

Conservation of Energy (if \( \dot{Q} = P = 0 \))

\[ \dot{E} = 0 \Rightarrow \Delta E = E_2 - E_1 = 0 \]  (IIIb)

If no energy flows into a system, then its energy does not change.

Statics (if \( \dot{E}_K \) is negligible)

\[ \dot{Q} + P = \dot{E}_p + \dot{E}_{\text{int}} \]  (IIId)

If there is no change of kinetic energy then the change of potential and internal energy is due to mechanical work and heat flow.

Pure Mechanics (if heat flow and dissipation are negligible)

\[ P = \dot{E}_K + \dot{E}_p \]  (IIId)

In a system well modeled as purely mechanical the change of kinetic and potential energy is due to mechanical work.
Some Definitions

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Notes</th>
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</thead>
<tbody>
<tr>
<td>( \mathbf{r} ) or ( \mathbf{x} )</td>
<td>Position</td>
<td>.e.g., ( \mathbf{r}<em>i \equiv \mathbf{r}</em>{i/O} ) is the position of a point i relative to the origin, ( O )</td>
</tr>
<tr>
<td>( \dot{\mathbf{r}} ) or ( \mathbf{v} )</td>
<td>Velocity</td>
<td>.e.g., ( \mathbf{v}<em>i \equiv \mathbf{v}</em>{i/O} ) is the velocity of a point i relative to ( O ), measured in a non-rotating reference frame</td>
</tr>
<tr>
<td>( \ddot{\mathbf{r}} ) or ( \mathbf{a} )</td>
<td>Acceleration</td>
<td>.e.g., ( \mathbf{a}<em>i \equiv \mathbf{a}</em>{i/O} ) is the acceleration of a point i relative to ( O ), measured in a Newtonian frame</td>
</tr>
<tr>
<td>( \mathbf{\omega} )</td>
<td>Angular velocity</td>
<td>A measure of rotational velocity of a rigid body</td>
</tr>
<tr>
<td>( \mathbf{\alpha} )</td>
<td>Angular acceleration</td>
<td>A measure of rotational acceleration of a rigid body</td>
</tr>
<tr>
<td>( \mathbf{L} )</td>
<td>Linear momentum</td>
<td>A measure of a system’s net translational rate (weighted by mass)</td>
</tr>
<tr>
<td>( \mathbf{\dot{L}} )</td>
<td>Rate of change of linear momentum</td>
<td>The aspect of motion that balances the net force on a system</td>
</tr>
<tr>
<td>( \mathbf{H}_C )</td>
<td>Angular momentum about point C</td>
<td>A measure of the rotational rate of a system about a point C (weighted by mass and distance from C)</td>
</tr>
<tr>
<td>( \mathbf{\dot{H}}_C )</td>
<td>Rate of change of angular momentum about point C</td>
<td>The aspect of motion that balances the net torque on a system about a point C</td>
</tr>
<tr>
<td>( E_K )</td>
<td>Kinetic energy</td>
<td>A scalar measure of net system motion</td>
</tr>
<tr>
<td>( E_{int} )</td>
<td>Internal energy</td>
<td>The non-kinetic non-potential part of a system’s total energy</td>
</tr>
<tr>
<td>( P )</td>
<td>Power of forces and torques</td>
<td>The mechanical energy flow into a system. Also, ( P = \dot{W} ), rate of work</td>
</tr>
<tr>
<td>( [I^{cm}] )</td>
<td>Moment of inertia matrix about cm</td>
<td>A measure of how mass is distributed in a rigid body</td>
</tr>
</tbody>
</table>

Please also look at the tables inside the back cover.
### Table 1

#### Momenta and energy

<table>
<thead>
<tr>
<th>What system</th>
<th>Linear Momentum</th>
<th>Angular Momentum</th>
<th>Kinetic Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>In General</td>
<td>$L = \frac{d}{dt}L$</td>
<td>$H_C = \frac{d}{dt}H_C$</td>
<td>$E_K = \frac{1}{2}m_0 \frac{d^2\vec{x}}{dt^2} + \frac{1}{2}(\vec{r}<em>{cm/C} \times \vec{\omega}) \cdot \vec{r}</em>{cm/C}$</td>
</tr>
<tr>
<td>One Particle $P$</td>
<td>$m_0 \vec{\dot{p}}$</td>
<td>$\vec{r}_{cf/C} \times \vec{\omega} m_0$</td>
<td>$\frac{1}{2}m_0 \frac{d^2\vec{p}}{dt^2}$</td>
</tr>
<tr>
<td>System of Particles</td>
<td>$\sum_{i=1}^{n} m_i \vec{\dot{q}_i}$</td>
<td>$\sum_{i=1}^{n} \vec{r}_{cf/C} \times \vec{\omega} m_i$</td>
<td>$\frac{1}{2} \sum_{i=1}^{n} \frac{d^2\vec{q}_i}{dt^2}$</td>
</tr>
<tr>
<td>Continuum</td>
<td>$\int \vec{\dot{u}} , dm$</td>
<td>$\int \vec{r}_{cf/C} \times \vec{\omega} , dm$</td>
<td>$\frac{1}{2} \int \frac{d^2\vec{u}}{dt^2} , dm$</td>
</tr>
<tr>
<td>System of Systems</td>
<td>$\sum_{i=1}^{n} \sum_{c_{i,c}} \vec{\dot{u}_i}$</td>
<td>$\sum_{i=1}^{n} \vec{\dot{H}_C_i}$</td>
<td>$\sum_{i=1}^{n} \sum_{c_{i,c}} \vec{H}_{C_i}$</td>
</tr>
</tbody>
</table>

#### Rigid Bodies

| Rigid Body | $m_0 \vec{\dot{x}_0}$  | $m_0 \vec{\dot{\theta}_0}$  | $\vec{r}_{cm/C} \times \vec{\dot{\omega}} m_0 + \frac{\vec{T}^m}{\vec{H}_{cm}} \vec{\dot{\omega}}$  | $\vec{r}_{cm/C} \times \vec{\dot{\omega}} m_0 + \frac{\vec{T}^m}{\vec{H}_{cm}} \vec{\dot{\omega}} \times \vec{H}_{cm}$  | $\frac{1}{2} m_0 \frac{d^2\vec{x}_0}{dt^2} + \frac{1}{2} \vec{\dot{\omega}} \cdot \vec{r}_{cm/C} \vec{\dot{\omega}}$ |
| 2D Rigid Body in $xy$ Plane with $\vec{\omega} = \vec{\omega} \vec{k}$ | $m_0 \vec{\dot{x}_0}$  | $m_0 \vec{\dot{\theta}_0}$  | $\vec{r}_{cm/C} \times \vec{\dot{\omega}} m_0 + \frac{\vec{T}^m}{\vec{H}_{cm}} \vec{\dot{\omega}}$  | $\vec{r}_{cm/C} \times \vec{\dot{\omega}} m_0 + \frac{\vec{T}^m}{\vec{H}_{cm}} \vec{\dot{\omega}} \times \vec{H}_{cm}$  | $\frac{1}{2} m_0 \frac{d^2\vec{x}_0}{dt^2} + \frac{1}{2} \vec{\dot{\omega}}^2 \frac{\vec{r}_{cm/C}^2}{\vec{H}_{cm}}$ |
| One Rigid Body with $\vec{\omega} = \vec{\omega} \vec{k}$ | $m_0 \vec{\dot{x}_0}$  | $m_0 \vec{\dot{\theta}_0}$  | $\vec{T}^m \cdot \vec{\dot{\omega}} = \vec{H}_{C}$  | $\vec{T}^m \cdot \vec{\dot{\omega}} \times \vec{H}_{C}$  | $\frac{1}{2} \vec{\dot{\omega}} \cdot \vec{T}^m \vec{\dot{\omega}}$ |
| 2D Rigid Body with $\vec{\omega} = \vec{\omega} \vec{k}$ | $m_0 \vec{\dot{x}_0}$  | $m_0 \vec{\dot{\theta}_0}$  | $\vec{T}^m \cdot \vec{\dot{\omega}}$  | $\vec{T}^m \cdot \vec{\dot{\omega}} \times \vec{H}_{C}$  | $\frac{1}{2} \vec{\dot{\omega}}^2 \frac{\vec{T}^m}{\vec{H}_{cm}}$ |

The table has used the following terms:

$m_0$ = total mass of system,

$m_i$ = mass of body or subsystem $i$,

$r_{cm/C}$ = position of the center of mass relative to point $C$,

$\vec{v}_i$ = velocity of center of mass of sub-system or particle $i$,

$\vec{a}_i$ = acceleration of center of mass of sub-system $i$,

$\vec{H}_{C_i}$ = angular momentum of subsystem $i$ relative to point $C$.

$\vec{H}_{C_i}$ = rate of change of angular momentum of sub-system $i$ relative to point $C$. 

$\vec{H}_{cm} = \sum r_{i/cm} \times (m_i \vec{\dot{v}_i})$ angular momentum about the center of mass

$\vec{H}_{cm} = \sum r_{i/cm} \times (m_i \vec{\dot{a}_i})$ rate of change of angular momentum about the center of mass

$\vec{\Dot{\omega}}$ is the angular velocity of a rigid body,

$\vec{\Dot{\omega}} = \vec{\dot{a}}$ is the angular acceleration of the rigid body,

$[\vec{T}^m]$ is the moment of inertia matrix of the rigid body relative to the center of mass, and

$[\vec{\omega}]$ is the moment of inertia matrix of the rigid body relative to a fixed point (not moving point) on the body.
1D motion / 1D forces / 1 or more particles

** Kinematics, 1D **

pos. \( x \)

vel. \( v = \dot{x} \)

\[ \frac{dx}{dt} \]

accel. \( a = \ddot{x} \)

\[ \frac{dv}{dt} = \frac{d^2x}{dt^2} \]

** \( LMB \) (Linear Momentum Balance), 1D **

\[ \sum F_x = ma \]

\[ \text{Net} = ma \]

All 1D problems reduce to:

\[ a = \frac{\text{Net}}{m} \]

from \( LMB \)

\[ \frac{dv}{dt} = \frac{\text{Net}}{m} \]

system of

\[ \frac{dx}{dt} = v \]

2 1st order ODE's

For \( 1 \), separate variables integrate

\[ \int_{v_0}^{v} dv = \int_{t_0}^{t} a(t^*) dt^* \; ; \; v_0 = v(t_0) \]
\[ v(t) = v_0 + \int_{t_0}^{t} a(t^*) \, dt^* \]

For (2) separate variables, integrate
\[ x(t) = x_0 + \int_{t_0}^{t} v(t^*) \, dt^* \]

\[ \text{Alternative: } a = \frac{dv}{dt} = \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx} \text{ chain rule} \]

\[ \Rightarrow \int_{x_0}^{x} v(x^*) \, dx^* = \int_{x_0}^{x} a(x^*) \, dx^* \]

\[ \Rightarrow \frac{v^2}{2} - \frac{v_0^2}{2} = \int_{x_0}^{x} a(x^*) \, dx^* \]

**Energy**

\[ E_K(\text{Kinetic Energy}) = \frac{1}{2} m v^2 \]

\[ E_K = m u a \]

\[ P_{\text{in}} = F_{\text{net}} v \]

\[ PB (\text{Power Balance}) \]

\[ P_{\text{in}} = E_K \]
Work \ W_{12} = \int_{x_1}^{x_2} F(x^*) \, dx^*

**Work-Energy Principle**

\[ W_{12} = \Delta E_k \]

**Work-Power**

\[ W_{12} = \int_{t_1}^{t_2} P(t) \, dt \]

**Potential Energy**

\[ E_p(x) = -\int_{x_0}^{x} F(x^*) \, dx^* \]

\[ F(x) = -\frac{dE_p}{dx} \]

**Conservation of Energy**

\[ E_T = E_p + E_k = \text{const} \]

\[ \Rightarrow E_T^0 \Rightarrow \Delta E_T = 0 \]
**System of Particles**

**Center of Mass**

\[ C_{\text{CM}} \]
\[ m_{\text{tot}} \times \text{CM} = \sum m_i \times x_i \quad \Rightarrow \quad m_{\text{tot}} = \frac{\sum m_i}{\sum m_i} \]
\[ x_{\text{CM}} = \frac{\sum m_i \times x_i}{m_{\text{tot}}} \]
\[ v_{\text{CM}} = \frac{\sum m_i \times v_i}{m_{\text{tot}}} \]
\[ a_{\text{CM}} = \frac{\sum m_i \times a_i}{m_{\text{tot}}} \]

\[ \text{System} \quad \text{CMBA} \quad \sum F_x = M_{\text{tot}} \quad a_{\text{CM}} \]
\[ = \sum m_i \times a_i \]

Also,
\[ \sum F_x = m_{\text{tot}} \left( L_x \right)_{\text{tot}} \]
\[ F_{\text{net}} = 0 \Rightarrow \left( L_x \right)_{\text{tot}} = 0 \]
\[ \Rightarrow \text{Conservation of Linear Momentum} \]
\[ \left( L_x \right)_{\text{tot}} = \text{const} \Rightarrow \]
\[ m_{\text{tot}} \left( v_{\text{CM}} \right) = \text{const} \Rightarrow v_{\text{CM}} = \text{const} \]
\[ E_k = \sum \frac{1}{2} m_i \left( v_i \right)^2 = \frac{1}{2} m_{\text{tot}} v_{\text{CM}}^2 \]
\[ E_p = \frac{1}{2} k \Delta s^2 \]

\[ F_s = k \Delta s; \Delta s = (x_2 - x_1) \]

Linear spring

Linear dashpot

\[ F_d = b \Delta \dot{d}; \Delta \dot{d} = (\dot{x}_2 - \dot{x}_1) \]

Linear drag

\[ F_d = -C \dot{V} \hat{e} \]

Quadratic drag

\[ F_d = -C |\dot{V}| \hat{e} \]

Gravity

\[ \vec{F} = -mg \hat{y} \]

\[ E_p = mg \hat{y}; E_p(0) = 0. \]

Simple Harmonic Motion (SHM)

\[ \ddot{x} = -C x; \text{SHM} \ (C > 0) \]

Solution: \[ x(t) = C_1 \cos \sqrt{C} t + C_2 \sin \sqrt{C} t \]

\( C_1, C_2 \) determined by initial conditions \( (I.C.'s) \), \[ x(0) = x_0, \dot{x}(0) = \dot{x}_0. \]

Radial freq. \( \equiv \sqrt{C} \); natural perl. \( \equiv 2\pi / \sqrt{C} \)