MATH 110: LINEAR ALGEBRA
MIDTERM #2 REVIEW SOLUTIONS

Problem 1.

(a) False. If one only uses only type 3 operations (see p. 148), then this is true; however, a type 1 operation (interchanging rows) changes the determinant by -1, and type 2 operations (scaling by a nonzero constant c) scale the determinant by c.

(b) True. If $AB = O$, then the columns of B are contained in the nullspace of A, therefore $\operatorname{rk}(B) \leq \operatorname{null}(A)$. By the rank-nullity theorem,

$$\operatorname{null}(A) + \operatorname{rk}(A) = n$$

so

$$\operatorname{rk}(A) + \operatorname{rk}(B) \leq n.$$

(c) Nonsense; the rows and columns are different!

What is true is that the *dimensions* of these spaces are the same, i.e. row rank is equal to column rank.

(d) False. If $c \in F$ is any constant, then cI is only similar to itself. That is to say, if P is any invertible matrix, then

$$P(cI)P^{-1} = c(PP^{-1}) = cI.$$

In fact, these are all such matrices!

(e) False, since the product of B ($n \times m$) with A ($m \times n$) yields an $n \times n$ matrix. Even the latter to an $n \times n$-identity matrix, it is also false. If A has rank m and is $m \times n$, then we must have $m \leq n$. But then in the sequence

$$F^n \xrightarrow{A} F^m \xrightarrow{B} F^n$$

we visibly see that the first map is not injective, therefore the composition cannot be injective, so there is no way that we could have $BA = I$.

What *is* true is that there exists an $n \times m$ matrix B such that $AB = I$, where I is the $m \times m$ identity matrix. Do you see how to prove this?

(f) Nonsense.

(g) This is true. We prove it by induction. For $n = 1$, we have

$$a + b = \frac{a^2 - b^2}{a - b}.$$

Suppose it is true for all values less than n; we prove it is true for n.

Expanding about the last row, we obtain

$$(-1)^{n+(n-1)} \det \begin{pmatrix} a+b & ab & 0 & \dots & 0 \\ 1 & a+b & ab & \dots & 0 \\ 0 & 1 & a+b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & ab \end{pmatrix}$$

$$+ (-1)^{n+n}(a+b) \det \begin{pmatrix} a+b & ab & 0 & \dots & 0 \\ 1 & a+b & ab & \dots & 0 \\ 0 & 1 & a+b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a+b \end{pmatrix}.$$

For the latter matrix, by induction it has determinant

$$\det A = \frac{a^n - b^n}{a-b}.$$

For the first matrix, we apply row expansion about the last row one more time, and we find by induction that it has determinant

$$ab\frac{a^{n-1} - b^{n-1}}{a-b}.$$

Putting this together, our matrix has determinant

$$-ab\frac{a^{n-1} - b^{n-1}}{a-b} + (a+b)\frac{a^n - b^n}{a-b} = \frac{-a^n b + ab^n + a^{n+1} + a^n b - ab^n - b^{n+1}}{a-b} = \frac{a^{n+1} - b^{n+1}}{a-b}.$$

(h) False, the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

has determinant zero, since the sum of the first two rows is equal to the sum of the last two rows, but the formula gives $1(1) - (-1)(1) = 2 \neq 0$.

(i) True. Suppose A is similar to B, so that $B = P^{-1}AP$ for some invertible matrix P. Then

$$\det(B - tI) = \det(P^{-1}AP - tI) = \det(P^{-1}AP - tPP^{-1}) =$$
$$= \det(P^{-1}(A - tI)P) = \det(P^{-1})\det(A - tI)\det(P) = \det(A - tI).$$

(j) True. In fact, A and A^t have the same characteristic polynomial! We have

$$\det(A - tI) = \det((A - tI)^t) = \det(A^t - tI).$$

(k) False. Consider the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Both have characteristic polynomial $(1-t)^2$. The matrix I is diagonal(izable), but A is not; if it were, since its only eigenvalue is 1, then it would be similar to the identity matrix, but it is only similar to itself!

(l) False, the matrix A in the previous part has only eigenvalues 1. The other direction, though, is true. If $A^2 = I$, and λ is an eigenvalue of A with eigenvector v then $Av = \lambda v$ so $A^2 v = \lambda^2 v = v$, so $\lambda^2 = 1$, so $\lambda = \pm 1$.

(m) True. The only eigenvalues of A are zero, since if λ is an eigenvalue with eigenvector v, then $A^k v = \lambda^k v = 0$, so $\lambda^k = 0$, so $\lambda = 0$. If A were diagonalizable, then it would be similar to the zero matrix, which is only similar to itself and hence A itself would be zero, a contradiction.

(n) False. The identity matrix satisfies the polynomial $t - 1 = 0$, but its characteristic polynomial is $(t-1)^2$.

(o) True. Suppose $f(t) = c_n t^n + \cdots + c_1 t + c_0$. If λ is an eigenvalue with eigenvector v, then

$$f(A)v = c_n A^n v + \cdots + c_1 A v + c_0 v = c_n \lambda^n v + \cdots + c_1 \lambda v + c_0 v = f(\lambda) v.$$

Therefore $f(\lambda)$ is an eigenvalue with eigenvector v.