MATH 110: LINEAR ALGEBRA
HOMEWORK #6 WORKSHEET SOLUTIONS

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Problem 1. Part (a) is true: the matrix P represents elementary row operations and the matrix Q represents elementary column operations. Part (b) is false: take A to be the 1×1 matrix 2. Then the matrix in the middle is the 1×1 identity matrix, hence P cancels with P^{-1}, and you have the impossible equation $2 = 1$. (There will be much more on this later, as it concerns diagonalization!) Statement (c) is false: Take

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\mathrm{rk}(A) = \mathrm{rk}(B) = 1$, but $A^2 = A$ and $B^2 = O$, so $\mathrm{rk}(A^2) = 1 \neq 0 = \mathrm{rk}(B^2)$. I leave statement (d) as a challenge! Part (e) is true: Write $B = B_{\mathrm{re}} + iB_{\mathrm{im}}$, where B_{re} and B_{im} have as entries the real and imaginary parts of B. Then

$$AB = A(B_{\mathrm{re}} + iB_{\mathrm{im}}) = AB_{\mathrm{re}} + iAB_{\mathrm{im}} = I$$

so $AB_{\mathrm{re}} - I = -iAB_{\mathrm{im}}$. But the matrix on the left has entries real numbers, and the matrix on the right has entries which are purely imaginary, so they must both be zero: hence $AB_{\mathrm{re}} = I$. Since inverses are unique, we have $B = B_{\mathrm{re}}$ as claimed. [One could also prove this by using complex conjugation: $\overline{AB} = A\overline{B} = I$, so $B = \overline{B}$, hence B is a real matrix.]

Problem 2. Let A_1 be the matrix obtained by adding the last row to each of the other $n-1$ rows; then $\det(A) = \det(A_1)$. Each entry of A_1 except for the last row is either $-2, 0, 2$. Let A_2 be the matrix obtained by dividing each of these rows by 2; then $2^{n-1}\det(A_2) = \det(A_1)$. The matrix A_2 has integer entries, so $\det(A_2)$ is an integer, hence $\det(A)$ is divisible by 2^{n-1}. [One could also prove this by induction: then just add the second row to the first to get a matrix A' and expand the determinant about the first row: one obtains

$$\det(A') = \det(A) = \sum_{i=1}^{n} a'_{1i} \det(\widetilde{A_{1i}})$$

where each a'_{1i} is either $-2, 0, 2$, and hence is divisible by 2. By induction, $\widetilde{A_{1i}}$ is a matrix with $1, -1$ entries, so it has determinant divisible by 2^{n-1}. Thus $\det(A')$ is the sum of n terms, each divisible by 2^n so $\det(A)$ itself is divisible by 2^n.]

Problem 3. We prove this by induction. The result is true for a 1×1-matrix, since then

$$\Delta_1 = \frac{b(p_1 - a) - a(p_1 - b)}{b - a} = p_1$$

which is the determinant of the matrix!

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Now assume the statement is true for all $k \leq n$. Add the second-to-last column to the last column and the second-to-last row to the last row. The determinant does not change, and one obtains the matrix

$$\begin{pmatrix} p_1 & a & a & \dots & a & 0 \\ b & p_2 & a & \dots & a & 0 \\ b & b & p_3 & \dots & a & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \dots & p_{n-1} & a - p_{n-1} \\ 0 & 0 & 0 & \dots & b - p_{n-1} & p_n \end{pmatrix}$$

Expand the matrix about the last row, to get

$$(-1)^{(n-1)+n}(b - p_{n-1}) \det \widetilde{A_{n-1,n}} + (-1)^{n+n} p_n \Delta_{n-1}.$$

The $(n-1) \times (n-1)$-matrix $\widetilde{A_{n-1,n}}$ has zeros in the last column except for the entry $a - p_{n-1}$ in row $n-1$ and column $n-1$, and by expansion about this column we obtain $\det(A_{n-1,n}) = (a - p_{n-1})\Delta_{n-2}$. Putting these together, we obtain

$$p_n \Delta_{n-1} - (b - p_{n-1})(a - p_{n-1})\Delta_{n-2}.$$

Now you just need to do some algebra to show that this is equal to Δ_n! [It's fun: exactly what you need cancels. OK, so this problem really tests your perservance!]

[If you're curious, there's also a very clever way to do this by considering the entries a, b as *variables*: then the determinant is a polynomial in a, b, which vanishes whenever $a = p_i$ or $b = p_i$...]