MATH 110: LINEAR ALGEBRA
HOMEWORK #4 WORKSHEET SOLUTIONS

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Problem 1. Let $x_1, \dots, x_n$ be an ordered basis for W_1 and $y_1, \dots, y_m$ be an ordered basis for W_2. Then we claim that $\beta = x_1, \dots, x_n, y_1, \dots, y_m$ is an ordered basis for V, and leave it as homework.

We claim that already $[T]_\beta$ has the desired form. Indeed, $T(x_i) \in W_1$, so $T(x_1)$ is a linear combination of the x_i, and in particular, all coefficients of y_j are zero. Similarly, $T(y_j) \in W_2$, so all of its coefficients for x_i are zero. Therefore the coefficient c_{ij} of $[T]_\beta$ is zero unless c_{ij} is in the upper-left $m \times m$ matrix A_1 or the lower-right $n \times n$ matrix A_2. Thus $[T]_\beta$ has the desired form.

For the last statement, consider the linear transformation associated to A_1, namely, $T_{A_1} : W_1 \to W_1$. It is invertible if and only if it is bijective if and only if it is onto, as required.

Problem 2. Suppose that $a_1T_1 + \cdots + a_nT_n = 0$ for some $a_i \in \mathbb{R}$. Then for all polynomials $f \in V$, we have

$$(a_1T_1 + \cdots + a_nT_n)(f) = 0(f) = 0.$$

In particular, this is true for $f(x) = x^n$. We see that

$$T_i(x^n) = (x^n)^{(i)} = n(n-1)\dots(n-i+1))x^{n-i}$$

so

$$(a_1T_1 + a_2T_2 + \cdots + a_nT_n)(x^n) = a_1nx^{n-1} + a_2n(n-1)x^{n-2} + \cdots + a_nn! = 0.$$

But each term in this expansion has a different degree, so we must have $a_1 = \cdots = a_n = 0$.

Problem 3. We see that $T^2 = T_0$ if and only if $T^2(x) = T(T(x)) = 0$ for all $x \in V$, which holds if and only if $T(x) \in N(T)$ for all $x \in V$, which holds if and only if $R(T) \subset N(T)$.

For the second part, let $v_1, \dots, v_m$ be an ordered basis for $N(T)$, and extend β by the replacement lemma to a basis $\beta = v_1, \dots, v_m, w_1, \dots, w_n$ for all of V. Then $T(v_i) = 0$ for all i so the first m columns of $[T]_\beta$ are zero. For the w_j, if we let

$$T(w_j) = a_1v_1 + \cdots + a_mv_m + b_1w_1 + \cdots + b_nw_n,$$

then since $T(w_j) \in R(T) \subset N(T)$, we see that $b_1 = \cdots = b_n = 0$, hence the bottom n rows of $[T]_\beta$ are zero. This is the desired form.

Problem 4. Let $\beta = v_1, \dots, v_n$ be a basis for $R(T)$. Then $T(\beta)$ generates $R(T^2)$, and since $\mathrm{rk}(T) = \mathrm{rk}(T^2) = \#T(\beta)$, by the replacement lemma we must have that $T(\beta)$ is linearly independent!

So suppose that $v \in R(T) \cap N(T)$. Then $v \in R(T)$, so we may write $v = a_1v_1 + \cdots + a_nv_n$ for $a_i \in F$. But $v \in N(T)$ so $T(v) = 0$, hence

$$T(v) = a_1T(v_1) + \cdots + a_nT(v_n) = 0.$$

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But $T(\beta)$ is linearly independent, so $a_1 = \cdots = a_n = 0$, so $v = 0$. Thus $R(T) \cap N(T) = \{0\}$.

Problem 5. This is essentially §2.1 #13 and #14, just with the language of isomorphisms.

Problem 6. Statement (a) is true, it was proved in class, if $\dim V = n$ then $V \cong F^n$, so $V \cong W$ if and only if $\dim V = \dim W$. Statement (b) is false, since 0 is not in the set. Statement (c) is false, a counterexample is given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Statement (d) is also false, one actually has

$$\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta.$$

Problem 7. Left to the reader!