9. Similar Δs, equilateral Δs, squares.

1. Similar Δs,
   Let us first recall a few facts from the earlier sections.

1) \( \arg(z_2 - z_1) = \) angle between vector \( \vec{z_1z_2} \) and the x-axis.

\[
\arg(z_4 - z_3) - \arg(z_2 - z_1) = \arg\left(\frac{z_4 - z_3}{z_2 - z_1}\right)
\]

= angle between \( \vec{z_1z_2} \) and \( \vec{z_1z_3} \) from \( z_1 \) to \( z_2 \) to \( z_3 \) to \( z_4 \).

2) \( \arg(z) = -\arg(-z) \).
   \( z \) = reflection of \( z \) wrt x-axis.

Then \( \Delta \vec{z_1z_2z_3} \) is the reflection of \( \Delta \vec{z_1z_2z_3} \) wrt x-axis.

2) Vector \( z_2 = z_1 \cdot e^{i\phi} \) is obtained from:
   a) initial vector \( z_1 \),
   b) a rotation by angle \( \phi \) and
c) multiplication of the length by \( \rho \).
Notation:

\[ \Delta z_1z_2z_3 \sim \Delta w_1w_2w_3 \]  
\( \Rightarrow \) (same orientation)

\[ \Delta z_1z_2z_3 \sim \Delta u_1u_2u_3 \]  
\( \Rightarrow \) (reverse orientation)

Since \( \Delta z_1z_2z_3 \sim \Delta z_1z_2z_3 \) (reverse orientation),

\[ \Rightarrow \Delta u_1u_2u_3 \sim \Delta z_1z_2z_3 \]  
(same orientation).

We will now derive a formula that expresses the fact that these two \( \Delta \)s have two proportional pairs of sides, and the angle between these sides are equal:

\[ \angle z_1 = \angle w_1 \quad \text{and} \quad \frac{|z_1z_2|}{|z_1z_3|} = \frac{|w_1w_2|}{|w_1w_3|}. \]

Indeed, we have \( \angle z_1 = \arg \left( \frac{z_1z_2}{z_1z_3} \right) = \arg \left( \frac{w_1w_2}{w_1w_3} \right) \),

and similarly \( \angle w_1 = \arg \left( \frac{w_1w_2}{w_1w_3} \right) \).

Then we have shown that in similar \( \Delta \)s,

\[ \frac{z_1z_2}{z_1z_3} = \frac{w_1w_2}{w_1w_3}, \quad \text{or} \]

\[ \frac{z_2-z_1}{z_2-z_1} = \frac{w_2-w_1}{w_3-w_1} \Rightarrow \]

\[(z_2-z_1)(w_3-w_1) - (w_2-w_1)(z_3-z_1) = 0, \quad \text{or} \]

\[ \left| \begin{array}{cc} z_2-z_1 & w_2-w_1 \\ z_3-z_1 & w_3-w_1 \end{array} \right| = 0 \iff \Delta z_1z_2z_3 \sim \Delta w_1w_2w_3 \]  
(same orientation).
Now let $\Delta z_1z_2z_3 \sim \Delta w_1w_2w_3$ (reverse orientation).

Then

$$\Delta w_1w_2w_3 \sim \Delta z_1z_2z_3 \text{ (same orientation)},$$

$$\Rightarrow \quad \begin{vmatrix} z_2-z_1 & w_2-w_1 \\ z_3-z_1 & w_3-w_1 \end{vmatrix} = 0 \Leftrightarrow \Delta z_1z_2z_3 \sim \Delta w_1w_2w_3 \text{ (rev. or.)}$$

Note: One can show that

$$\begin{vmatrix} z_2-z_1 & w_2-w_1 \\ z_3-z_1 & w_3-w_1 \end{vmatrix} = \begin{vmatrix} z_1 & w_1 \\ z_2 & w_2 \\ z_3 & w_3 \end{vmatrix} = \begin{vmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$
Ex. 1. On each side of an arbitrary quadrangle, draw a square externally. Show that the two segments joining the centers of the opposite squares are perpendicular to each other and of equal length.

Soll. 1) \( u_1 - z_1 = i(z_2 - z_1) \Rightarrow \) vector \( z_2 \), rotated by \( \frac{\pi}{2} \)

\[ u_1 = z_1 + i(z_2 - z_1) \]

\[ u_2 = \overrightarrow{z_2} + i(z_2 - z_1) \]

\[ \Rightarrow C_1 = \frac{1}{2} (u_1 + z_2) = \frac{1}{2} \left( \frac{\text{Diag. of } \triangle ABC}{2} \right) = \frac{z_1 + z_2 + i(z_2 - z_1)}{2} \quad (1) \]

Similarly,

\[ C_2 = \frac{z_2 + z_3 + i(z_3 - z_2)}{2} \quad (2) \]

\[ C_3 = \frac{z_3 + z_4 + i(z_4 - z_3)}{2} \quad (3) \]

\[ C_4 = \frac{z_4 + z_1 + i(z_1 - z_4)}{2} \quad (4) \]

Thus

We need to show that \[ \frac{c_3 - c_1}{c_4 - c_2} = i \]

\[ c_3 - c_1 = \frac{z_3 + z_4 + i(z_4 - z_3) - z_1 - z_2 - i(z_2 - z_1)}{2} = \frac{(z_3 + z_4 - z_1 - z_2)}{2} + i(z_4 + z_3 - z_2) \]

\[ c_4 - c_2 = \frac{z_4 + z_1 + i(z_1 - z_4) - z_2 - z_3 - i(z_3 - z_2)}{2} = \]

\[ = \frac{(z_4 + z_1 - z_2 - z_3) + i(-z_4 - z_2 + z_1 + z_3)}{2} = \]

\[ = -i \left( \frac{i(z_4 + z_1 - z_2 - z_3) + \frac{1}{2}(z_1 + z_2 - z_3 - z_4)}{2} \right) = -i \left( c_3 - c_1 \right), \checkmark \]
(3) Equilateral triangles.

In (1) and (2) we found the criteria which allow us to tell whether two \( \Delta \)s are similar and whether a given quadrangle is a square, respectively.

We'll now obtain a mathematical criterion of whether a given \( \Delta \) is equilateral.

Preliminaries

In an equilateral \( \Delta \), angles are 60°, \( \Rightarrow \) we need a "nice" expression for 60°, so far we only have an expression for 120° = \( 2\pi/3 \) = \( \omega \).

But note that

\[
e^{i \cdot 60^\circ} = e^{i \pi/3} = (e^{i \cdot 2\pi/3})^{1/2} = \omega^{1/2}.
\]

On the other hand,

\[
e^{i \pi/3} = e^{i \left( \frac{4\pi}{3} - \pi \right)} = e^{i \frac{4\pi}{3}} \cdot e^{-i\pi} = e^{i \pi} \cdot (-1) = -\omega^2.
\]

(See the picture above.)

Thus, the mathematical expression is:

\[
e^{i \pi/3} = -\omega^2.
\]
Now consider an equilateral \( \triangle z_1z_2z_3 \)

Note that its vertices are listed in the counterclockwise order, which is the same order in which the cubic root of 1, \( 1, \omega, \omega^2 \), appear.

In an equilateral \( \triangle \), \( \overrightarrow{z_1z_3} = (\overrightarrow{z_2} \text{ rotated by } 60^\circ) \)

\[
(\overrightarrow{z_3} - \overrightarrow{z_1}) = (\overrightarrow{z_2} - \overrightarrow{z_1}) \cdot (-\omega^2) = e^{-i\pi/3}
\]

\[
z_1(-1-\omega^2) + z_2 \cdot \omega^2 + z_3 \cdot \omega^3 = 0
\]

\[
\omega \overrightarrow{z_1} + \omega^2 \overrightarrow{z_2} + \omega^3 \overrightarrow{z_3} = 0
\]

or equivalently

\[
1 \cdot \overrightarrow{z_1} + \omega \cdot \overrightarrow{z_2} + \omega^2 \cdot \overrightarrow{z_3} = 0
\]

This is the criterion that \( \triangle z_1z_2z_3 \) is equilateral.

If \( z_1, z_2, z_3 \) are oriented clockwise, then we simply switch the order of \( \omega \)'s to: \( \omega^2, \omega, 1 \) (or \( \omega^3, \omega^2, \omega \)). Thus, in this case:

\[
\omega^2 \overrightarrow{z_1} + \omega \overrightarrow{z_2} + \overrightarrow{z_3} = 0
\]
Ex. 2 Napoleon's Thm.

On each side of an arbitrary $\triangle z_1 z_2 z_3$, construct a triangle equilateral $\Delta$ externally. Then the centroids of these new equilateral $\Delta$s are the vertices of the 4th equilateral $\Delta$.

Given: $\triangle z_1 z_2 z_3$ = arb.
$\Delta w_1 z_2 z_3$ etc. = equil
$C_{1,2,3}$ = centroids
Want: $\Delta c_1 c_2 c_3$ = equilat.

Proof:

1) $\Delta w_1 z_2 z_3$ = equilat. $\Rightarrow$

$$w_1 + \omega z_2 + \omega^2 z_3 = 0 \quad \text{(note the order!)}$$

$$w_1 = -\omega z_2 - \omega^2 z_3.$$  \hspace{1cm} (1)

Note: We want to solve for $w_1$. Therefore, we chose its coefficient to equal 1.

Similarly,

$w_3 + \omega z_1 + \omega^2 z_2 = 0 \Rightarrow$

$$w_3 = -\omega z_1 - \omega^2 z_2.$$  \hspace{1cm} (2)

$w_2 + \omega z_3 + \omega^2 z_1 = 0 \Rightarrow$

$$w_2 = -\omega z_3 - \omega^2 z_1.$$  \hspace{1cm} (3)
2) \( C_1 = \text{centroid of } \Delta w_1 z_2 z_3 \Rightarrow \)

\[
C_1 = \frac{1}{3} (w_1 + z_2 + z_3) = \frac{1}{3} \left[ (-\omega z_2 - \omega^2 z_3 + z_2 + z_3) \right] = \frac{1}{3} \left( (1-\omega)z_2 + (1-\omega^2)z_3 \right). \tag{4}
\]

Similarly:
- \( C_2 = \frac{1}{3} (w_2 + z_3 + z_1) = \frac{1}{3} \left( (1-\omega)z_3 + (1-\omega^2)z_1 \right) \tag{5} \)
- \( C_3 = \frac{1}{3} (w_3 + z_1 + z_2) = \frac{1}{3} \left( (1-\omega)z_1 + (1-\omega^2)z_2 \right) \tag{6} \)

3) It remains to show that \( \omega^0 c_1 + \omega c_2 + c_3 = 0 \) (note the order!)

LHS. = \[
\left[ \frac{1}{3} \left( \omega^2 (1-\omega)z_2 + \omega^2 (1-\omega^2)z_3 + \omega (1-\omega)z_3 + \omega (1-\omega^2)z_1 \right) + (1-\omega)z_1 + (1-\omega^2)z_2 \right]
\]

\[
= \frac{1}{3} \left[ 2\omega - \omega^3 + 1 - \omega \right] + \frac{1}{3} \left[ \omega^2 - \omega^3 + 1 - \omega^2 \right] + \left[ \frac{1}{\omega} \omega^2 + \omega - \omega^2 \right] = 0. \quad \checkmark
\]

q.e.d.