Sec. 8. Review of useful facts in plane geometry.

1. Facts (with some proofs).

(F1) a) \( l_1 \parallel l_2 \Rightarrow \angle \alpha_1 = \angle \alpha_2 \) (corresponding angles).

Thus, if \( l_1 \parallel l_2 \), the following diagram takes place:

\[ \hat{\alpha}_2 (= \alpha_2) \text{ and } \alpha_2 \text{ are called vertical angles.} \]

\[ \alpha + \beta = 180^\circ. \]

b) The sum of the angles in any \( \Delta \) is \( 180^\circ \).

Proof:

(F2) Criteria of equal triangles.

a) Two \( \Delta \)s are equal iff any of the 3 criteria below hold.

The following respective elements of the two triangles are equal:
SAS: 2 adjacent sides and the angle between them.

ASA: 1 side and 2 angles adjacent to it.

SSS: 3 sides.

b) Two right triangles are equal iff:

RLH: The leg of one Δ equals one of the legs of the other Δ, and their hypotenuses are equal.

Note: This does not hold if the two Δs are not the right Δs.

F3 Similar Δs.

a) Two triangles are similar iff the 3 angles of one Δ equals the respective angle of the other Δ.

\[ \triangle ABC \sim \triangle A_1B_1C_1. \]

Note: The order of the vertices is important: \( \triangle ABC \neq \triangle B_1A_1C_1 \), etc.
b) If \( \triangle ABC \sim \triangle A_1B_1C_1 \) \& \( \triangle A_1B_1C_1 \sim \triangle A_2B_2C_2 \) \( \Rightarrow \triangle ABC \sim \triangle A_2B_2C_2 \).

c) If \( \triangle ABC \sim \triangle A_1B_1C_1 \) \( \Rightarrow \frac{AB}{A_1B_1} = \frac{BC}{B_1C_1} = \frac{AC}{A_1C_1} \)
\& \( \frac{AC}{BC} = \frac{A_1C_1}{B_1C_1} \), \( \frac{AC}{AB} = \frac{A_1C_1}{A_1B_1} \), \( \frac{AB}{BC} = \frac{A_1B_1}{B_1C_1} \).

By definition, \( \text{ABCD} \) is called a \underline{parallelogram} if its opposite sides are \underline{parallel}. Then:

(a) In a parallelogram, \( \alpha + \beta = 180^\circ \) (see picture).

(b) In a parallelogram, the opposite sides are \underline{equal}.

(c) The converse of (b) is true:

A quadrilateral whose opposite sides are equal, is a \underline{parallelogram}.

(proof at home.)

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F5  Thales Theorem

\( (l_1 || l_2) \iff \left( \frac{a}{p} = \frac{b}{q} \right) \)

F6  \underline{Isosceles triangles}

Def: A \( \triangle \) \( \text{is called isosceles} \) if two of its sides are \underline{equal}.

In an isosceles \( \triangle \):

(a) Two angles adjacent to the base are \underline{equal}.
b) The median, altitude, and the angle bisector drawn to the base of an isosceles \( \triangle \) all coincide with the perpendicular bisector to the base. 

E.g., let's prove that the angle bisector is also a median.

Know: \( \angle \text{ABM} = \angle \text{CBM} \); Want: \( |\text{AM}| = |\text{MC}| \).

Indeed: \( \triangle \text{ABM} = \triangle \text{CBM} \) by \([F2a]\) (SAS), because 
\( |\text{AB}| = |\text{BC}| \) and \( |\text{BM}| \) is the common side. Then 
\( |\text{AM}| = |\text{MC}| \) as the respective sides in equal \( \triangle \)s.

(F7) Let \( \text{OC} \) be the bisector of an arbitrary angle \( \angle \text{LO} \). Then the distances from any point \( \text{C} \) on \( \text{OC} \) to the sides of \( \angle \text{LO} \) are equal.

Know: \( 4 \alpha = \beta \) (see picture), \( |\text{AC}| = |\text{CB}| \).

Proof: \( \angle \text{AOC} = \angle \text{BOC} \) (given), \( \angle \text{OAC} = \angle \text{OBC} = 90^\circ \) (by constr.) \( |\text{OC}| \) is common hypothesis \( \Rightarrow \) by \([F2b]\), \( \triangle \text{OAC} = \triangle \text{OBC} \)
\( \Rightarrow |\text{CA}| = |\text{CB}| \) as respective sides.

Converse: Prove at home.

(F8) Let \( \ell \) be the perpendicular bisector of a segment \( \text{AB} \). Then:

\( (\text{C} \in \ell) \iff (|\text{CA}| = |\text{CB}|) \)

Proof: at home.
Let $\triangle ABC$ be a right triangle, and $\angle B = 90^\circ$, and let $h$ be the altitude drawn from vertex $B$.

Then \[
\frac{h}{p} = \frac{a}{h} \implies h^2 = pq
\]

**Proof:** Home.

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**F10** Law of cosines:

In any $\triangle ABC$,

\[c^2 = a^2 + b^2 - 2ab \cos \gamma\]

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**F11** The area of any triangle can be found as:

\[S = \frac{1}{2} ah = \frac{1}{2} a \cdot b \sin \gamma\]

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**F12** Let $l$ be the tangent line to the circle at point $A$. Then $OA \perp l$.

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**F13** Let a central angle $\angle ACR = 2\alpha$, where $O$ is the center of the circle.
Then any angle subtended by arc AB (i.e. the angle $\angle ACB$, where $C$ is any point on the larger arc $AB$) equals $\alpha$.

Proof:

**Case (i)**

1) $\triangle BOC$ is isosceles because $|BO| = |OC|$ are the radii of the circle.
   Hence by (F6a), $\angle OCB = \angle OBC$.

2) $\angle AOB = 180^\circ - \angle BOC \quad (\text{as complementary})
   \quad \frac{2\alpha}{2}$

But $(\angle OCB + \angle OBC) = 180^\circ - \angle BOC \quad (\text{by F16}).$
Since $\angle OCB = \angle OBC \quad (\text{by i}), =$
$2 \angle OCB = 2\alpha, \Rightarrow \angle OCB = \alpha$, q.e.d.

**Case (ii)** Draw $CD$ s.t. $O \in CD$. Then by case (i),
$\angle ACD = \frac{1}{2} \angle AOB \Rightarrow \angle DCB = \frac{1}{2} \angle DOB$

$\Rightarrow \angle ACB = \angle ACD + \angle DCB = \frac{1}{2}(\angle AOD + \angle DOR) =
\quad \frac{1}{2} \angle AOB, \quad q.e.d.$
Case (iii) At home.

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b) In particular, any angle subtended by a diameter is the right angle:

\[ \angle \text{right angle} \]

\[ \angle \text{right angle} = 90° \]

c) Let ABCD be a quadrangle inscribed into a circle.

Then \( \angle A + \angle C = 180° \),
\[ \angle B + \angle D = 180°, \]
or
\[ \angle A = 180° - \angle C \]
\[ \angle B = 180° - \angle D \]

Proof: At home. Hint:

\[ \text{Diagram} \]

2. Basic constructions by the straightedge and a compass.

One can construct a \( \perp \) through a given pt. to a given line.

C1 a) Given line \( l \) and point \( P \) \( \notin l \), construct \( m \perp l \) at \( P \) \( \in m \).
Solution:

\[ A, B: \mid PA \mid = \text{arbitrary} \]
\[ P': r = \text{arbitrary} \]
\[ (\geq AB/2). \]

Then \( PP' \perp e \).
\[ \Rightarrow PP' = m. \]

(see (E8)).

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\( \ell \) Same but \( P \in e \):

Solution:

\[ A, B: \mid PA \mid = \text{arbitrary} \]
\[ Q, Q': r (\geq |PA|) = \text{arbitrary}. \]

Then
\[ QQ' \perp e, \text{ and} \]
\[ PE \subseteq QQ', \Rightarrow \]
\[ QQ' = m. \]

\( \Box \) Given \( P \notin e \), one can construct \( m \parallel e \) s.t. \( p \in m \).

Solution:

\[ A: PA = r = \text{arbitrary} \]
\[ B: AB = r \]
\[ C: \angle ABC = r \& PC = r \]
\[ (C = @ @ r \cap @ r). \]

Thus \( APCR = \Box \text{ rhombus} \Rightarrow PC \parallel AB = e \).
(C3) a) Given a segment $AB$, one can construct any integer multiple of $AB$:

$$
\begin{array}{c}
A \quad B \quad C \quad D \quad \cdots \\
\hline
BC = AB \\
CD = AB, \\
etc.
\end{array}
$$

b) Given two segments of length $a$ and $b \leq a$, one can construct a segment of length $(a - b)$.

$$
\begin{array}{c}
a \\
A \quad C \quad E \quad B
\end{array}
AB = a \\
BC = b \\
AC = a - b.
$$

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(C4) a) One can divide any segment into two equal parts.

Solution: (based on [ES])

$Q, Q'$: $r = \text{arbitrary}$

$QQ' \perp AB$, and

$QQ' \cap AB = M$,

where $|AM| = |MB|$.

b) One can divide any segment into the ratio $m : n$, where $m, n$ are any two natural numbers.

Solution: Let segment $\overline{AB}$ be given:

$$
\begin{array}{c}
A \quad B \\
C \\
\hline
|CA| = a, \\
B \cap \ell = 0.
\end{array}
$$
Given 3 segments of length $a, b, c$, one can construct the segment $x = a \frac{b}{c}$.

Solution: Construct $p \parallel q$ (by $\Box$).

\[ x = \sqrt{a \cdot b} \]  
($\Leftarrow x^2 = a \cdot b$).

Solution:
1) Construct $AC = a + b$  
(C3a)

2) $M =$ midpoint of $AC$  
(C4a)
   Draw a circle with center $M$ and radius $MA$.

3) $l \perp AC$ and $B \in l$ (Use $C16$).
   $l \cap$ circle = $D$

4) $\angle ADC = 90^\circ$  
(F13b), $\Rightarrow |DB|^2 = |AB| \cdot |BC|$  
($F9$).

$|DB| = \sqrt{a \cdot b}$.
Given \( \alpha \) and \( \beta < \alpha \),

\[
x = \sqrt{\alpha^2 + \beta^2} \quad \text{and} \quad y = \sqrt{\alpha^2 - \beta^2}
\]

Solution: At home.

(C7) a) One can construct an angle equal an integer multiple of a given angle.

Solution (only for doubling an angle).

Let \( \angle AOB \) be the given angle; \( A, B \) - arbitrary points on its sides.

1) Circle \( (O; \alpha) \) \( \cap \)

Circle \( (B; \frac{\alpha}{2}) = C \).

2) Then \( |OC| = |CA|, |BC| = |BA| \Rightarrow \)

\( \triangle OCB = \triangle OAB \) by \( \text{SSS} \) \( \text{(F2a)} \).

Then \( \angle COB = \angle AOB \). ✓

b) One can bisect a given angle.

Solution:

\( A, B \) - \( (O; r) \) marked

\( C \) - \( (A; r) \) marked \( (r > \frac{1}{2}) \)

\( \text{Finish at home} \)

supply the necessary justifications) ✓
Note: One cannot trisect an arbitrary angle using only the straightedge and a compass.

C8) One can construct a tangent line that passes through a given point not inside the circle.

a) The point is on the circle.

Given \((F12)\), the construction reduces to \((C16)\).

b) The point is outside the circle.

Solution: Let \(A\) be the given point.

1) \(M = \text{midpoint of OA (by C9a)}\)

2) Construct a circle with the center \(A\) and radius \(|MO| = |MA|\).

3) The intersection point of this circle with the given circle is \(B\).

Then \(AB\) is tangent to the given circle.

At home, supply the necessary details of the proof.

C9) One can construct angles:

a) \(90^\circ\) (see \((C4)\));

b) \(60^\circ\);

c) \(45^\circ = \frac{1}{2} \times 90^\circ\) (see a), \((C7b)\); d) \(30^\circ = \frac{1}{2}\) (see d), \((C7d)\).
3. Center of a triangle.

3a. Centroid = \[ G = \frac{1}{3} (A + B + C) \]

The centroid is the intersection of the medians of a triangle (Ex. 4 in Lec. 4).

If \( A, B, C \) = complex numbers, then

**Construction:**

Construct midpoints \( A_1, B_1, C_1 \) and then construct \( A_{\overline{1}}, B_{\overline{1}}, C_{\overline{1}} \).

3b. Circumcenter

Consider an arbitrary \( \triangle ABC \):

Let \( A_1, B_1, C_1 \) be the midpoints of the sides opposite to \( A, B, C \).
Let \( l_A, l_B, l_C \) be the perpendicular bisectors to \( BC, AC, \) and \( AB \), respectively.

1) Let \( O = l_A \cap l_B \).

Since \( OC \perp l_A \Rightarrow |OC| = |OB| \) \( \ell \) (by (E8)).

\[ O \in l_B \Rightarrow |OA| = |OC| \]

Hence \( |OC| = |OB| = |OC| \) \( \ell \) (A)

2) But since \( |OC| = |OB| \), \( \Rightarrow O \in l_C \) (by (E8)),
\( \Rightarrow \) the three perpendicular bisectors always intersect at 1 point.

By (A), this point is the center of the circle passing through \( A, B, C \). Name: circumcircle of \( \triangle ABC \).
3c] Incenter: Consider an arbitrary $\triangle ABC$ and the interior angles.

![Diagram of an incenter with labels and angles]

1) Let $O = M_A \cap M_B$ and $OA_1, OB_1, OC_1$ be the perpendiculars from $O$ to the sides $BC, AC, AB$. Then $OA_1, OB_1, OC_1$ are the distances from $O$ to these sides.

2) By (7), $O \in M_A$ (bisector of $\angle BAC$)

\[ |OC_1| = |OB_1|, \]

$O \in M_B \Rightarrow |OC_1| = |OA_1|.$

Thus $|OC_1| = |OA_1| = |OB_1|$ \hspace{1cm} (44)

3) Since $|OB_1| = |OA_1|$, $\Rightarrow$ by the converse of (7),

\[ O \in M_C, \Rightarrow \text{ the interior angle bisectors of a triangle meet at one point.} \]

Because of (44), $O$ is the center of a circle inscribed into $\triangle ABC$ (it is tangent to each of its sides). \hspace{1cm} [E12] 

3d] Orthocenter:

![Diagram of an orthocenter with labels and altitudes]

Let $AA_1, BB_1, CC_1$ be the altitudes in an arbitrary $\triangle ABC$. Then they (or the lines containing them) meet at one point.

Proof: see p. 171. We will also prove this later in the course.