Sec. 21. The area in hyperbolic geometry.

1. General formula in UHP

As we said at the beginning of Sec. 20, it is technically easier to consider the area in the UHP model than in the disk model.

**Reason:** In UHP model, the integration is done in rectangular coordinates, while in disk model, the integration is done in polar coordinates.

\[ \Delta A = \Delta x \Delta y \]

But, according to the end of Sec. 20,

\[ \Delta x = \frac{\Delta x}{y}, \quad \Delta y = \frac{\Delta y}{y} \]

\[ \Rightarrow \Delta A = \frac{\Delta x \cdot \Delta y}{y^2} \]

Therefore, the formula for the area in the UHP model is:

\[ A_{\text{UHP}} = \iint_{\text{region}} \frac{dx dy}{y^2} \]

2. Areas of Triangles.

b. Doubly asymptotic \( \Delta \).
Disk model:

Generic UTEP model

Standard position in UTEP model:

\[ C \to \infty \]

Explanation of the angles:

\( B'Q' \) is tangent to h.s.l. \( B'A' \).
\( O \) is the Euclidean center of h.s.l. \( B'A' \).

Then \( \angle O B'Q' = \pi/2 \).
If \( \angle C'B'Q' = \alpha \), \( \Rightarrow \)
\( \angle O B'D' = \pi - \alpha - \frac{\pi}{2} = \frac{\pi}{2} - \alpha \),
\( \Rightarrow \angle BOD' = \frac{\pi}{2} - (\frac{\pi}{2} - \alpha) = \alpha \).

Also, if we take \( |OA'| = r \), then
\[ |OD'| = -r \cos \alpha \]
\[ A' = r \]
\[ D' = -r \cos \alpha \].
Then the area of the dashed region is:

\[ A = \iint_{\Delta} \frac{\text{d}x \text{d}y}{y^2} = \int_{x_{\min}}^{x_{\max}} \left( \int_{y_{\min}(x)}^{y_{\max}(x)} \frac{\text{d}y}{y^2} \right) \text{d}x \]

\[ = \int_{-r \cos \alpha}^{r} \left( \int_{\frac{r^2-x^2}{r^2}}^{\infty} \frac{\text{d}y}{y^2} \right) \text{d}x \]

\[ = \int_{-r \cos \alpha}^{r} \left[ -\frac{1}{y} \right]_{\frac{r^2-x^2}{r^2}}^{\infty} \text{d}x \]

\[ = \int_{-r \cos \alpha}^{r} \frac{\text{d}x}{\sqrt{r^2-x^2}} \]

\[ = \int_{-r \cos \alpha}^{r} \text{d}x \left| \begin{array}{c}
X = r \cos \Theta \\
\text{d}x = r \sin \Theta \text{d}\Theta \\
\sqrt{r^2-x^2} = r \sin \Theta
\end{array} \right.
\]

\[ = \int_{\Theta_{\min} = \pi - \alpha}^{\Theta_{\max} = 0} \frac{-r \sin \Theta \text{d}\Theta}{r \sin \Theta} = -\Theta \]

\[ \theta_{\min} = \pi - \alpha \]

Thus, we have shown that the hyperbolic area of a doubly asymptotic \( \Delta \) is:

\[ A = \pi - \alpha \]

where \( \alpha \) is the angle at the non-ideal vertex (the angles at the ideal vertices are both 0).
Remark 1: The area, as earlier the distance, in hyperbolic geometry is related to the angular measure.

Remark 2: The area of a triply-asymptotic $\triangle$ equals $\pi$.

**Generic UHP:**

**Standard Position UHP:**

(The Euclidean area is infinite, but the hyperbolic area is finite, due to $\frac{1}{y^2}$ in the integrand.)

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An arbitrary $\triangle$.

**Standard Position**

$A_1 = \infty$

$B_1 = AC \cap R$

$C_1 = AB \cap R$

$A_1 = BC \cap R (= \infty)$

Now connect $A_1, B_1, C_1$ pairwise by...
Then

\[ \Delta A_{ABC} = \Delta A_{A_1B_1C_1} + \Delta A_{A_1B_1C} + \Delta A_{ABC} + \Delta A_{A_1B_1C_1} \]

\[ \Delta A_{A_1B_1C_1} = \pi \] (see Remark 2 above)

\[ \Delta A_{A_1B_1C} = \pi - (\pi - \beta) = \beta \]

\[ \Delta A_{ABC} = \pi - (\pi - \gamma) = \gamma \]

\[ \Delta a_{A_1B_1C_1} = \pi - (\pi - \alpha) = \alpha \]

Then

\[ \Delta A_{ABC} = \Delta A_{A_1B_1C_1} - (A_{\Delta A_1B_1C} + A_{\Delta A_1B_1C} + \Delta A_{A_1B_1C_1}) \]

\[ = \pi - (\alpha + \beta + \gamma) \]

\[ \Delta A_{ABC} = \pi - (\alpha + \beta + \gamma) \]

where \( \alpha, \beta, \gamma \) are the angles of the hyperbolic \( \Delta ABC \).

As we know from \#6(ii) of HW30, the sum of the angles in the hyperbolic \( \Delta \) is less than \( \pi \). The difference is called the defect of the \( \Delta \).

Thus, we showed that the area of a hyperbolic \( \Delta \) equals its defect.
Remark 3: The same relations for the areas holds in the disk model.

3 Areas of rectangle quadrilaterals

\[ \begin{align*}
A_{ABCD} &= A_{ABD} + A_{BDC} = \pi - (\alpha + \beta + \delta_1) + \\
&\quad + \pi - (\beta_2 + \gamma + \delta_2) = \\
&\quad = 2\pi - (\alpha + (\beta_1 + \beta_2) + \gamma + (\delta_1 + \delta_2)) \\text{ & } \\
A_{ABCD} &= 2\pi - (\alpha + \beta + \gamma + \delta) \\text{ & } \\
\end{align*} \]

Corollary: The following equivalent of the Fifth Postulate is false in hyperbolic geometry:

These exist rectangles.

In other words, rectangles do not exist in hyperbolic geometry.

Indeed, in a rectangle, \( \alpha = \beta = \gamma = \delta = \pi/2 \). Then

\[ \text{A rectangle} = 2\pi - 4 \cdot \frac{\pi}{2} = 0, \]

i.e., it would be just a point!
Ex. 1: The hyperbolic area of a quadrilateral with 4 asymptotic vertices equals $2\pi$.

(all angles are zero!)

Disk:

Ex. 2: The hyperbolic area of Saccheri quadrilaterals can be between $0$ and $\pi$:

$$0 \leq A_{\text{Saccheri}} < \pi$$

Disk:

(these pictures in UHP aren't as obvious)

$$A = 2\pi - \left( \frac{\pi}{2} \cdot 2 + \alpha \cdot 2 \right) = \pi - 2\alpha.$$

$\alpha = 0$:

If the S.Q. is tiny, $\alpha \approx \frac{\pi}{2}$ (but still < $\frac{\pi}{2}$).
4. Absence of similar $\triangle s$ in hyperbolic geometry.

The fact that $\angle ABC = \pi - (A + B + C)$ entails yet another remarkable corollary.

In hyperbolic geometry, there can be no similar $\triangle s$.

Recall that one of the equivalents of the Fifth Postulate in Euclidean geometry said that there exist similar $\triangle s$.\( \text{[Footnote]} \)

Thus, we will prove the theorem. If in hyperbolic $\triangle pqr$ and $\triangle p'q'r'$ the corresponding angles are equal, then the triangles are congruent.

**Proof**

1) Apply a $T$ to place $r$ and $r'$ at $z=0$, and $p$ and $p'$, $q$ and $q'$ on the same respective radii (since $\angle prq = \angle p'r'q'$).

Then there are the following distinct possibilities:

(a) [Diagram]

(b) [Diagram]

(c) [Diagram]

(d) $p=p'$, $q=q'$, $r=r'$.

(\( \triangle pqr \cong \triangle p'q'r' \))
2) (a) is impossible because
\[ A_{p'q'r'} < A_{pqr}, \]
but \[ A_{p'q'r'} = \pi - (\angle p' + \angle q' + \angle r') = \]
\[ = \pi - (\angle p + \angle q + \angle r) = A_{pqr}, \]
which is a contradiction.

Similarly, (c) is impossible.

3) \hspace{1cm} \textbf{Let us show that (b) is also impossible.}

We recall that it is given that
\[ \angle rqp = \angle rq'p' \text{ (e.g., )}. \]
But then \[ \angle sqq' = \pi - \alpha, \]
and the sum of the angles of \[ \triangle sqq' \] is:
\[ \frac{\angle sqq'}{\pi - \alpha} + \frac{\angle sqq}{\alpha} + \frac{\angle sqq'}{\alpha} > \pi, \]
which is impossible in hyperbolic geometry.

Thus, case (b) is also impossible.

Case (d) is then the only possibility, \( \Rightarrow \)

\textit{Q.E.D.}
5 Euclidean geometry as a limiting case of hyperbolic geometry.

Recall #4 of HW 11:
The circumference of a hyperbolic circle is:

\[ C = 2\pi \sinh R, \]
where \( R \) is the hyperbolic radius.
Also, when \( R \ll 1 \), \( \sinh R \approx R \),

\[ C_{\text{hyp}} \approx 2\pi R. \]

In other words, when the radius of a hyperbolic circle is very small, the relation between its circumference and radius is almost the same as in Euclidean geometry.

Recall Ex. 2 in subsection 3.
Soccer quadrilaterals become almost rectangles when they are tiny.

These results lead one to state the following:
Over very short distances, the hyperbolic figures behave almost as Euclidean ones.
This is also corroborated by the fact that very short circular arcs are approximately straight segments.

E.g., this hyperbolic $\Delta$ looks almost like a Euclidean $\Delta$, and its angular sum is $\approx \pi$, and so the area is very close to 0.

(6) Tesselations in hyperbolic geometry.

In Euclidean geometry, convenient building blocks are rectangles. In hyperbolic geometry, rectangles do not exist. What can one use instead of them as building blocks?

Answer:

Quadrangles with

four

skipped