Instructions: Present your work in a neat and organized manner. Please use either the 8.5 × 11 size paper or the filler paper with pre-punched holes. Please do not use paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but not by folding its (upper left) corner.

You must show all of the essential details of your work to get full credit. If I am forced to fill in gaps in your solution by using nontrivial (at my discretion) steps, I will reduce your score for that particular assignment.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting your own work.

All problems in a given assignment contribute equal amount (1 point) to the assignment’s total score, unless otherwise noted.

Homework Assignment # 7
Due Monday, October 21, 2013

1. In this problem, you will re-obtain the result of Problem 1 of HW # 5. There, you obtained that result by means of elementary geometry. Here you will use complex numbers and algebraic calculations.

This result will be extensively used in Part 2 of the course.

(i) Consider a unit circle with the center, O, at the origin. Let point A = a with 0 < a < 1 be on the real axis (and, of course, inside the circle). Then point B = 1/a is also on the real axis. You need to prove that any circle that passes through A = a and B = 1/a is perpendicular to the unit circle. You may follow the steps outlined below.

Step 1 On which line must the center of any circle passing though two given points be located? Explain your answer by referring to (one of the) basic facts listed in Lec. 8.

Step 2 Let the center of a circle passing through the A and B specified above be denoted by C. Let the vertical coordinate of C be denoted by y. Using your answer in Step 1 and also the Pythagorean theorem, calculate the radius (e.g., |CA|) of this circle. Also find the distance |OC|. Your answers should be in terms of a and y.

Step 3 Let one of the two points where this circle intersects the unit circle be denoted as D. Use your answers to Step 2 to show that $|OD|^2 + |DC|^2 = |OC|^2$.

Step 4 The latter fact implies that $\triangle ODC$ is a right triangle. Explain why this means that the two circles are perpendicular. Refer to the definition of perpendicular circles found in Problem 1 of HW # 5 and to one or more of the basic facts.

Finally, note that you have proved the following result:

Let O be the center of a unit circle. Any circle that passes through the two points A and B such that $|OA| \cdot |OB| = 1$ and where O, A, and B are collinear, is perpendicular to the unit circle.

(ii) Now prove the converse of the above fact. Namely, let C be a circle that is perpendicular to the unit circle. The center of circle C is at some point C, and the radius is arbitrary. Let D be one of the intersection points of these two circles.

Draw any line through O that intersects circle C at two points: point A inside the unit circle and point B outside it. If A and B are not on the real line, we can rotate the entire coordinate system to place A and B on the real line. This will not restrict our consideration because such a rotation will not change relative angles and distances. Since A and B are now on the real line, let $A = a$ and $B = \tilde{a}$ (so, $a, \tilde{a} \in \mathbb{R}$).

Your goal is to prove that $\tilde{a} = 1/a$.

As in part (i), express |OC| and the radius of circle C through the coordinate(s) of point C and the numbers a and $\tilde{a}$. Next, from the fact that the two circles are perpendicular, explain why $\triangle ODC$ is a
right triangle. (That is, quote a part of Problem 1 of HW 5 which proved this.) Then apply to \( \triangle ODC \) the Pythagorean theorem to obtain \( \tilde{a} = 1/a \).

Finally, state the “if and only if” statement you obtained by combining the results of parts (i) and (ii).

In addition, draw a picture illustrating this theorem and then, using the notations of that picture, restate your “if and only if” statement in an easy-to-read form like this: \( \text{statement } A \leftrightarrow \text{statement } B \) (where you need to fill in the statements \( A \) and \( B \)). This will help you to do the next problem.

2. (2 points)

A new result derived in this problem will also be extensively used in Part 2 of the course.

(i) Consider the unit circle with the center, \( O \), at the origin. Also consider two points, \( A \) and \( B \), which are not on the unit circle. To be specific, let us assume that \( |OA| \equiv a < 1 \) and \( |OB| \equiv b < 1 \).

Show that there exists a unique circle that passes through both \( A \) and \( B \) and is also perpendicular to the unit circle.

Note that what you need to do here is:
(i) Show that such a circle can be constructed and
(ii) Show that no other such a circle can be constructed.

You may follow the steps outlined below. Without loss of generality (see part (ii) of Problem 1), assume that \( A \) is on the real axis. Then in the generic case, which you will consider now, \( B \) is not on the real axis. (The other, nongeneric, cases are considered in part (ii) below.)

Step 1 If the required circle passes through \( A \) and is perpendicular to the unit circle, what other point in the complex plane must it also pass through? (Use the result of Problem 1.) For further reference, let us call that point \( A' \).

Step 2 How many points uniquely determine a circle? (Refer to Lec. 8.3b or Lec. 10.1.) What does this imply for the problem in question? Once you have correctly answered the last two questions, you have shown that the required circle can be constructed.

Step 3 You have not yet shown that this circle is unique, for the following reason. What if in Step 1, you ask the question about point \( B \) instead of point \( A \)? Then you proceed to Step 2 and see that one can have two circles: one contains \( A, A', \) and \( B \), and the other contains \( A, B, \) and \( B' \) (the last point is defined analogously to \( A' \)). You then need to show that these two circles coincide.

Hint: Probably the easiest way to do so is to again use a result from Problem 1(ii). According to it, if \( C_{ABA} \) is a circle perpendicular to the unit circle and going through point \( B \), then what other point must this circle go through? An alternative way would be to show that points \( A, B, A', \) and \( B' \) are cocyclic, by computing their cross ratio. If you are not sure what complex number corresponds to \( B' \), find a related result in the posted solution to Problem 5(ii) of HW 3.

(ii) In part (i), you considered the generic situation where neither \( A \) nor \( B \) are on the unit circle and where \( B \) is not collinear with \( O \) and \( A \). Now describe how your answer to part (i) will modify in each of the following three cases.

(a) \( O, A \) and \( B \) are collinear, but \( A \neq B \). In this case, you need to distinguish two subcases, which will differ by the number of circles one can draw. State this number for each subcase.

Note for (a): Recall from Lec. 10.3 that a straight line is a circle of infinite radius.

(b) \( O, A \) and \( B \) are not collinear, \( A \) is on the unit circle, and \( B \) is inside the unit circle.

(c) \( O, A \) and \( B \) are not collinear, and both \( A \) and \( B \) are on the unit circle.

Hint for (c): The constructions used for part (i) and Cases (a,b) will not work here. Instead, using one of the basic facts and the definition of perpendicular circles, you can find the center of the required circle.

3. This problem can be done using several different methods. However, I want you to do it using the method shown below. Since the idea is for you to practice applying this method, I will not give any credit for a solution obtained by a different method.
Let triangle $\triangle ABC$ be inscribed into a unit circle with the center at the origin. As in Lec. 12.2, denote $A = a^2$, $B = b^2$, $C = c^2$, with $|A| = |B| = |C| = 1$. At the end of Lec. 12.2, we obtained the following expression for the incenter of such a triangle: $I = -ac + ab + bc$. We also showed that the points of intersection of the angle bisectors of $\triangle ABC$ with the unit circle (see the figure on the right) are given by the following expressions: $A_1 = bc$, $B_1 = -ac$, $C_1 = ab$. Thus, $I = A_1 + B_1 + C_1$, which suggests that $I$ is the orthocenter of $\triangle A_1B_1C_1$ (see Lec. 11.1c). You are now asked to prove this fact directly, by showing that $AA_1 \perp B_1C_1$. (The other two perpendicularity relations will follow from the latter relation by symmetry.) To do so, use the technique illustrated by the following example.

**Example** Let $|\alpha| = |\beta| = 1$. Prove that $(\alpha - \beta)/(\alpha + \beta)$ is purely imaginary. (This was done in Lec. 5, but here our focus is to illustrate a different method.)

**Proof:** Since $|\alpha| = |\beta| = 1$, we can let $\alpha = \exp[i\phi_\alpha]$ and $\beta = \exp[i\phi_\beta]$ for some (real) angles $\phi_\alpha$ and $\phi_\beta$. Then:

$$
\frac{\alpha - \beta}{\alpha + \beta} = \frac{\exp[i\phi_\alpha] - \exp[i\phi_\beta]}{\exp[i\phi_\alpha] + \exp[i\phi_\beta]} = \frac{\exp[i(\phi_\alpha + \phi_\beta)/2](\exp[i(\phi_\alpha - \phi_\beta)/2] - \exp[i(\phi_\beta - \phi_\alpha)/2])}{\exp[i(\phi_\alpha + \phi_\beta)/2](\exp[i(\phi_\alpha - \phi_\beta)/2] + \exp[i(\phi_\beta - \phi_\alpha)/2])}
$$

$$
= \frac{\exp[i(\phi_\alpha - \phi_\beta)/2] - \exp[-i(\phi_\alpha - \phi_\beta)/2]}{\exp[i(\phi_\alpha - \phi_\beta)/2] + \exp[-i(\phi_\alpha - \phi_\beta)/2]} = \frac{2i\sin[(\phi_\alpha - \phi_\beta)/2]}{2\cos[(\phi_\alpha - \phi_\beta)/2]} = i \cdot \frac{\sin[(\phi_\alpha - \phi_\beta)/2]}{\cos[(\phi_\alpha - \phi_\beta)/2]},
$$

which is purely imaginary.

**Hint:** In the problem in question, you will have to guess (or find by trial and error) what expression, instead of $\exp[i(\phi_\alpha + \phi_\beta)/2]$, needs to be factored out of both the numerator and denominator. The following observation may help. Look, e.g., at the denominator of the expression in the Example. How is the exponent of the factored term related to the exponents of the two original terms?

**Bonus** (Credit will be given only if the solution is more than 50% correct.)

(i) **(1 point)** Let $3\alpha$, $3\beta$, $3\gamma$, and $R$ be the angles and the circumradius, respectively, of an arbitrary triangle $\triangle ABC$. (Note that unlike the notations of Lec. 12.3, here $\angle BAC = 3\alpha$, $\angle ABC = 3\beta$, $\angle ACB = 3\gamma$.) Show that the side of the Morley’s triangle (i.e. the equilateral triangle obtained by intersection of adjacent angle trisectors) equals $8R \sin \alpha \sin \beta \sin \gamma$.

**Note:** To get credit, you must give a proof based on complex numbers.

**Hint:** You may follow these steps. First, express the side, say $\lambda_2\lambda_3$, as a product of three complex factors. Second, relate $b^3$ and $c^3$ to the angles defined above. For that, use basic facts $\text{[F1a]}$ and $\text{[F1b]}$. Third, use the technique introduced in Problem 3 to obtain the desired answer.

(ii) **(0.75 point)** Obtain the expression for $\lambda_1$ given at the end of Lec. 12.

**Hint:** Use a trick similar to the one used when transforming $(b^3 - 1)$ in the derivation of $\lambda_3$. Also, recall that $1/\omega = \omega^2$, or, equivalently, $\omega = 1/\omega^2$. **Note:** The goal of this problem is to make you use that trick. Thus, no credit will be given if you obtain an expression for $\lambda_1$ in the form $\lambda_1 = A/B$ and then prove that $\lambda_1B = A$.

(iii) **(0.25 point)** Verify the very last equation of Lec. 12 and thereby provide the final stroke to the derivation of Morley’s Theorem. Credit will be given only if you first mark like terms (different like terms must be marked differently), then combine their coefficients, and finally show that all those combined coefficients vanish.