Project 3:

Concrete application of bases

Goal

Apply the concept of bases in $\mathbb{R}^n$ to a practical problem.

General requirements

- You may work alone or with one other person. If you work with someone else, hand in one answer sheet with both of your names on it.
- No groups bigger than two. No collaboration between groups.
- Write your answers on the answer sheet provided in the last few pages of this document. Staple all the paper showing your neatly presented work to the answer sheet.

Introduction

As you know, a basis $\{v_k\}_{k=1}^n$ in any vector space is such a set of vectors that:

any vector $b$ in this space can be represented as a linear combination of the basis vectors,

$$b = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n,$$

and none of the basis vectors are redundant (i.e., can be replaced by some linear combination of the other basis vectors). The coefficients $b_k$ are called coordinates of vector $b$ in basis $\{v_k\}_{k=1}^n$. They show “how much” each of the basis vectors “contributes” into $b$.

An example of a basis that you are well familiar with is the basis of unit coordinate vectors in the plane or in the 3D space. Coordinates in such a basis are the usual vector coordinates you are also well familiar with. Bases naturally appear in many practical engineering applications as well. For example, any sound signal can be represented as a linear superposition of (many) sinusoidal waves with closely spaced frequencies. These sinusoidal waves, called Fourier harmonics, form a basis in the space of all signals with “reasonable” shape. The coordinates of any sound wave in this basis are the strengths of the corresponding Fourier harmonics.

In Exercise 3 of Project 1 you encountered another example of a basis. There, nonfat milk, soy flour, and whey formed a basis of ingredients from which a diet with any desired content of protein, carbohydrates, and fat could be created. The amounts of nonfat milk, soy flour, and whey in any diet can be viewed as “coordinates” of this diet in the basis of these three ingredients. In this Project, you will study in detail a mathematically similar problem occurring in construction business rather than in nutritional science.

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1 Your grade will be reduced by 5% if you hand in a pile of non-stapled sheets.
2 I will reduce your grade by an amount left to my discretion in each particular case if your work is presented in a messy way and I have to waste time deciphering it.
3 Well, not really any. Some restrictions do apply — see Parts (e) and Bonus-(b) of this Project. However, for our immediate purposes, we will still use the word ‘any’.
Problem description

Concrete mixes, used, e.g., for constructing sidewalks and building bridges, are composed of five main materials: cement, water, sand, gravel, and fly ash. By varying the percentages of these materials, mixes of concrete can be produced with differing characteristics. For example, the water to cement ratio affects the strength of the final mix, the sand to gravel ratio affects the “workability” of the mix, and the fly ash to cement ratio affects the durability. Different jobs require concrete with different characteristics. Now, preparing concrete “from scratch”, i.e. from the aforementioned five “primordial” ingredients, is time-consuming. Therefore, building supply companies tend to prepare, in the morning of each business day, some basic mixes from which they will be able to quickly create custom mixes for their customers during the business hours.

You are hired as a summer intern by a building supply company. On your first day of work, the company’s manager informs you that his company currently stocks three basic mixes of concrete with the following characteristics:

<table>
<thead>
<tr>
<th>Ingredient (type)</th>
<th>Super-strong (type S)</th>
<th>All-purpose (type A)</th>
<th>Long-life (type L)</th>
</tr>
</thead>
<tbody>
<tr>
<td>cement</td>
<td>20 lb</td>
<td>18 lb</td>
<td>12 lb</td>
</tr>
<tr>
<td>water</td>
<td>10 lb</td>
<td>10 lb</td>
<td>10 lb</td>
</tr>
<tr>
<td>sand</td>
<td>20 lb</td>
<td>25 lb</td>
<td>15 lb</td>
</tr>
<tr>
<td>gravel</td>
<td>10 lb</td>
<td>5 lb</td>
<td>15 lb</td>
</tr>
<tr>
<td>fly ash</td>
<td>0 lb</td>
<td>2 lb</td>
<td>8 lb</td>
</tr>
</tbody>
</table>

The manager then asks you to use your knowledge of Linear Algebra to give answers to the following questions within one week. (The Hints from your Linear Algebra instructor, whom you have contacted for help, are listed after the questions.)

(a) Does one actually need to stock all three of these mixes, or is any one of them redundant?

(b) Can one get all possible custom mixes from the above three basic ones? If ‘yes’, explain why. If ‘no’, explain why and describe all possible custom mixes that one can get from them.

(c) A customer has requested 3000 lb of a custom mix with the following proportions of cement, water, sand, gravel, and fly ash: 16, 10, 21, 9, 4. Find the amounts of type S, type A, and type L mixes needed to create this custom mix. Is the solution unique?

(d) A new competitor company in the area has announced that it stocks the following four (instead of only three) basic mixes: type SA, which is an equal mixture of types S and A; type SL, which is an equal mixture of types S and L; type AL, which is an equal mixture of types A and L; and a new type N, which has the following proportions of cement, water, sand, gravel, and fly ash: 10, 20, 5, 15, 10. Determine whether this other company’s claim about their being able to produce a larger variety of custom mixes is true in general (i.e., from the abstract Linear Algebra perspective).

(e) The customer mentioned in Part (c) is a regular customer of the company you are interning for, and he routinely orders large amounts of the custom mix described in Part (c). Therefore, it is very important to determine whether the new company mentioned in Part (d) is a serious competitor for this customer. Your task is to give an answer to this question.
The following questions will not (at least directly) affect the operation of the company, but the
manager has been curious about them for a while. So he asked you to answer them for an extra com-
pensation if you have the time.

(Bonus–(a)) Explain why it is always the case that water makes up one sixth the weight of any
custom mix, sand and gravel together make up half the weight, and cement and fly ash together make
up the remaining third.

(Bonus–(b)) Is it possible to invent a custom mix that satisfies the conditions in Bonus-(a) but
cannot be created from the three basic mixes? If ‘yes’, invent one. If ‘no’, explain why it is not
possible.

Hints

General: You can represent any mix by a vector \([c, w, s, g, f]^T\) (with the obvious meaning of \(c, w,\)
etc., and with ‘T’ standing for the transpose). The basic mixes can be represented by vectors \(S, A,\) and
\(L,\) whose components are given in the Table above. Then any custom mix that it is possible to create
out of the basic mixes can be written as their linear combination; see Introduction, and you may also
recall what you did in Ex. 3 of Project 1.

You are encouraged to perform algebraic manipulations with matrices, such as transforming them
into Reduced Echelon Form, using software. As long as all matrix entries are numbers (as opposed to
variables like \(x\)), you may use appropriate Matlab commands (see Project 1). You will not really have
to use software to manipulate a matrix with variable entries except in part Bonus-(b). Yet, if you decide
to do so, you may find the required commands in the example Mathematica file posted alongside this
Project. **Remember to attach printouts with computer commands you used and the answer(s) given by
the software.**

For Part (a): This is the material of the second subsection of Section 3.4 of the textbook.
For Part (b): This is the material of the first subsection of Section 3.4 of the textbook.
For Part (c): Set up and solve an appropriate linear system of the form of Eq. (1).
For Part (d): This is similar to Parts (a) and (b).
For Part (e): This is similar to Part (c), and you also need to use some common sense.

For Bonus–(a): (The following is just one possible way to answer this question.) Suppose you
mix \(x_1\) buckets of type \(S, x_2\) buckets of type \(A,\) and \(x_3\) buckets of type \(L\) (each bucket weighs 60
lb). What is the total weight of this mix? What is the total weight of water in it? Extend this to the
combinations of sand and gravel and of cement and fly ash.

For Bonus–(b): Answer this question in two steps. First, give an abstract answer where you should
ignore the specific nature of this problem and treat \(S, A, L\) as vectors in \(\mathbb{R}^5,\) which you can multiply
by arbitrary coefficients. Your abstract answer should come out to be ‘no’. Second, take the specific
nature of the problem into account by realizing that those coefficients should satisfy some (obvious)
restriction. With this restriction, show that the answer to the manager’s question should be ‘yes’.

One possible way to answer the first (abstract) question is to determine if the following two bases
span the same set of vectors. (You will need to explain why this will answer the question.) One basis
is the minimal spanning set which you obtained in Part (a) for \(\text{Sp}(\{S, A, L\}).\) The other basis can
be determined from the conditions stated in Bonus–(a) similarly to what is done in Exercises 1–8 for Section 3.4 of the textbook (see also Examples 3 and 5 of Section 3.4 and the beginning of Section 3.5.) Now that you have found these two bases, you want to determine whether they actually span the same subspace of $\mathbb{R}^5$. This can be done similarly to the following example, which refers to sets of two vectors, but can be straightforwardly generalized as needed. Suppose you have two vector sets \{a_1, a_2\} and \{b_1, b_2\}. They span the same subspace of $\mathbb{R}^n$ provided that any vector $x$ from $\text{Sp}\{a_1, a_2\}$, i.e., $x = x_1a_1 + x_2a_2$ with arbitrary $x_1$ and $x_2$, can also be represented as $x = y_1b_1 + y_2b_2$ for some $y_1$ and $y_2$; and vice versa. To determine if this is the case, follow the lines of Examples 1 and 2 in Section 3.4. Also, use Mathematica to carry out these calculations; a sample Mathematica notebook accompanies this Project. At the end of this calculation, your answer to the first, abstract, question, should be ‘no’ (as always, you need to explain why).

Now for step 2: Take the specific nature of the problem into account and show that the answer to your manager’s question is ‘yes’. Try to invent a mix in which one (or more) of some of the primordial ingredients is absent.

**Remark**

By comparing your answers for Parts (d) and (e) you could have gotten a feeling that the abstract answers provided by Linear Algebra may sometimes be irrelevant to real-life situations. (If you have also answered the Bonus questions, this feeling might have become even stronger.) That is, suppose there exist restrictions on coordinates which (the restrictions) are not equations but inequalities. Then the Linear Algebra treatment, based on solving linear systems of equations, may yield a solution that is not feasible. This is, of course, true. However, there is an entire scientific field that focuses on obtaining feasible solutions in such situations. It is called Linear Programming; it has many prominent applications in economics and other fields.

**Acknowledgement**

(a, 21 points) Is any of the three mixes redundant?

(b, 14 points) Yes or No? Why?

(c, 22 points) List the required amounts, in pounds, of mixes S, A, L. Is this answer unique?

(d, 19 points) In abstract Linear Algebra terms, can the new company produce a larger variety of custom mixes than your company?

(e, 24 points) Is the new company a serious competitor for the customer in question?

For Bonus questions, credit will be given only if your solutions are more than 50% correct and, especially for the first question, if the answer is stated coherently.

(Bonus–(a), 9 points) Attach additional pages with your work.

(Bonus–(b), 25 points) For step 1, attach pages with your work and a Mathematica printout. For step 2, give your example of the custom mix here and briefly explain why it serves its purpose.