Project 2:

Using linear systems for numerical solution
of boundary value problems

Goal

Introduce one of the most important applications of Linear Algebra to Engineering.

General requirements

- You may work alone or with one other person. If you work with someone else, hand in one answer sheet with both of your names on it.
- No groups bigger than two. No collaboration between groups.
- Write your answers on the answer sheet provided in the last few pages of this document. Staple\textsuperscript{1} all the paper showing your neatly presented\textsuperscript{2} work to the answer sheet.

Introduction

Most phenomena in Physics and Engineering are modeled by differential equations. You were introduced to simple differential equations of the form \( \frac{dy}{dx} = f(x, y) \) in Calculus II; perhaps, you also took a course (MATH 230 or MATH 271) focused on methods for solving such equations. A boundary value problem (BVP), whose approximate solution is illustrated in this Project, is just a differential equation supplied with two boundary values for the unknown variable \( y \). BVPs occur in a great variety of applications, such as (i) oscillation of membranes and strings (think of music!), (ii) sagging of beams and other structures under mechanical load, (iii) radiation by atoms and molecules, and (iv) hurricane modeling, to name just a few.

Most BVPs arising in modern applications of science and engineering are so complex that they cannot be solved exactly with pen and paper. Then one has to resort to solving them approximately on a computer. There is a whole area of applied mathematics which deals solely with this kind of solution of differential equations. It is called Numerical Analysis, and Linear Algebra lies at its foundation. Numerical analysis is very much driven by the need to solve practical problems: There are quite a few hi-tech companies which develop commercial software for solving differential equations on a computer.

In this Project, you will learn how to apply your knowledge of linear systems to solving one simple BVP. Linear systems enter this problem from two different angles. First, you will set up and solve a linear system to find an approximate formula for the second derivative of a function. Second, you will use that formula to set up another linear system, which describes the numerical solution of the BVP. Then you will use MATLAB to solve the latter linear system on a computer.

\textsuperscript{1}Your grade will be reduced by 5\% if you hand in a pile of non-stapled sheets.
\textsuperscript{2}I will reduce your grade by an amount left to my discretion in each particular case if your work is presented in a messy way and I have to waste time deciphering it.
Problem description and method of solution

A flexible power cable has one end staked to the ground and the other end fastened $H$ ft up a pole $L$ ft away, as shown in the left figure above. Let $y$ be the cable’s elevation at a point located $x$ horizontal units from the stake. Furthermore, let $W(x)$ be the vertical load density (i.e., cable’s weight per unit length) at a point $(x, y)$, and let $T$ be the (constant) tension of the cable. Assuming that both $W(x)$ and $T$ are known, the cable’s elevation can be shown to satisfy a second-order differential equation

$$\frac{d^2y}{dx^2} = \frac{W(x)}{T}. \quad (1)$$

Since the ends of the cable are pinned at specific locations, the following boundary conditions must supplement the differential equation:

$$y(0) = 0 \quad \text{and} \quad y(L) = H. \quad (2)$$

The above equations (1) and (2) form a BVP for the cable’s elevation, $y = y(x)$, as a function of the distance, $x$, from the stake.

The BVP (1) & (2) is sufficiently simple so that one can solve it exactly using just pen and paper. However, as you read in the Introduction, most differential equations of practical interest cannot be solved exactly. To solve them approximately, one has to use an approach that we illustrate below for the simple BVP (1) & (2). This approach ultimately relies on Linear Algebra.

The key idea behind finding an approximate solution to BVP (1) & (2) is this. Instead of looking for the cable’s elevation, $y(x)$, for all values of $x$ within the interval $[0, L]$, limit your goal to finding $y$ only at discrete values of $x$ on that interval. To this end, let $h = L/n$ and consider discrete points

$$x_i = ih, \quad \text{where} \; i = 0, 1, \ldots, n. \quad (3a)$$

This is illustrated in the right figure above for $n = 11$.\(^3\) Note that

$$x_0 = 0 \quad \text{and} \quad x_n = L. \quad (3b)$$

\(^3\)Incidentally, notice that the points $x_i$ subdivide the interval $[0, L]$ into $n$ subintervals of length $h = L/n$: $[0, h]$, $[h, 2h]$, $[2h, 3h]$, \ldots, $[(n-1)h, L]$. This may look familiar: You saw exactly the same approach when you first learned definite integration and Riemann sums in Calculus I and later learned various applications of it, such as finding areas and volumes, in Calculus II.
Further, denote
\[ y(x_i) = y_i. \] (3c)

So now, finding an approximate solution to the BVP (1) & (2) will amount to finding only the finitely many values \( y_i \), rather than the continuous function \( y(x) \) for all values of \( x \) on \([0, L]\).

Clearly, if \( h \) is taken to be very small, there is, practically, almost no difference between the continuous variables \( x \) and \( y \), on one hand, and the discrete sets \( \{x_0, x_1, x_2, \ldots, x_n\} \) and \( \{y_0, y_1, y_2, \ldots, y_n\} \), on the other. Yet, this discretization trick will allow you to replace the BVP given by Equations (1) and (2), which you do not know how to solve, by a linear system, which you know how to solve. The steps of transition from the BVP to a linear system are described below.

**Step-by-step instructions**

1. Your goal in this part is to derive an approximate formula for the second derivative of \( y(x) \), appearing in Equation (1), that involves only the discrete values \( y_i \) (\( i = 0, 1, \ldots, n \)). Follow the steps described below.

   To begin, denote (ah, again...) a point \( x_i = a \). Then it can be shown (just believe this) that the sought formula must have the form:
   \[
   y''(a) \approx A_{-1}y(a - h) + A_0y(a) + A_1y(a + h),
   \] (4)
   
   where \( A_{-1}, A_0, A_1 \) are constant coefficients that you need to determine. For examples of finding \( A_{-1}, A_0, A_1 \), you may consult either pp. 88–89 of the text book or/and Appendix A found after these Instructions.

   Write your formula (i.e., Equation (4) with your numerical values for the coefficients \( A_{-1}, A_0, A_1 \)) on the answer sheet and attach to it the paper with your derivation. Check your answer by comparing it with the solution of Exercise 21 of Section 1.8.

2. We are now going to return from the notation “\( a \)” to the notation “\( x_i \)”. We need to introduce yet a few more related notations. From the definition of \( x_i \) (see Equation (3a) above), we have:

   \[ a - h = x_i - h = ih - h = (i - 1)h = x_{i-1}. \]

   Accordingly, since we have denoted \( y(a) = y(x_i) = y_i \), then

   \[ y(a - h) = y(x_{i-1}) = y_{i-1}. \]

   Use these observations to rewrite the approximate formula you have obtained for \( y''(a) \) (that is, for \( y''(x_i) \)) in terms of \( y_{i-1}, y_i, \) etc.

   Next, at each interior point \( x_i \) of the interval \([0, L]\) (i.e., when \( 0 < i < n \)), approximate the differential equation (1) using the approximate formula for \( y''(x_i) \) that you have just written down. To this end, simply substitute that formula into the left-hand side of Equation (1). On the right-hand side of (1), use similar notations, i.e., \( W(x_i) = W_i \). Write the resulting equation at the point \( x_i \) in the answer sheet.
Thus, the outcome of this step should have the form

\[ (\ )y_{i-1} + (\ )y_i + (\ )y_{i+1} = (\text{something known}), \]

and your job is to supply the coefficients or expressions inside the parentheses.

3. Here you will practice setting up a linear system to approximate Equation (1) by using a small number \( n \) of discretization points. Later you will need to use a large \( n \).

Draw the interval \([0, L]\), subdivide it into 5 subintervals, and label the resulting points \( x_0, x_1, \ldots, x_5 \) (see Equations (3a) and (3b)).

Write your equation (5) for \( i = 1 \). Then repeat this for each interior point \( x_i \) of \([0, L]\). Note that each of your equations couples the value \( y_i \) to the values \( y_{i-1} \) and \( y_{i+1} \). Whenever you require the values \( y_0 \) and \( y_n \), these are found from the boundary conditions (2). Thus, the equations at the interior points \( x_i, i = 1, \ldots, 4 \) form a linear system for the values \( y_i \). Write this linear system for the unknown values \( y_1, y_2, y_3, y_4 \) in a matrix form. (Note that \( W_i \) and \( T \) are known in this problem.) Use the notation \( h \) for \( L/5 \).

4. Set up and solve a linear system (5) for the case \( n = 20 \) using MATLAB (see Appendix B after these Instructions). Use the values \( L = 100 \) ft, \( H = 12 \) ft, \( T = 3000 \) lb, and

\[
W(x) = \begin{cases} 
6 \text{ lb/ft} & \text{iced cable}, \quad 0 \leq x \leq 50 \text{ ft} \\
3 \text{ lb/ft} & \text{clean cable}, \quad 50 < x \leq 100 \text{ ft}.
\end{cases}
\]

Write the numerical values of your solution at several locations along the cable, as requested in the answer sheet. Round these values to 4 decimal places. Also, use MATLAB to plot the graph \( y = y(x) \) which you obtained on the interval \( 0 \leq x \leq L \) (i.e., include the end points). Label the axes. For example, to label the x-axis, type:

\[
\text{xlabel('whatever you want to use for the x-label','fontsize',12)}
\]

Print your computer output, including the plot of \( y(x) \), and staple it to the answer sheet.

Extra credit (partial credit is given only if the solution is mostly correct)

Consider a broken power line, where one end of the cable is still fastened to the pole but the other (free) end lies on the ground. At the point of contact with the ground, the cable’s slope is zero, and therefore the condition

\[ y'(0) = 0 \]

must be added to the boundary conditions (2). (Note that in this case, the tension \( T \) at the point of contact of the cable with the ground is provided solely by the static friction between the cable and the ground.) The resulting BVP, consisting of Equations (1), (2), and (6), has a solution only for one particular value of \( L \). This \( L \) is the distance from the pole to the point where the broken cable first touches the ground. Find \( L \) by setting up and solving a linear system for this case. Use the values: \( T = 500 \) lb, \( W(x) = 6 \) lb/ft (assume the cable is uniformly iced), and the other parameters as before. Approximate the first derivative in the boundary condition (6) by the formula

\[ y'(a) \approx \frac{y(a+h) - y(a)}{h}. \]
For definiteness, use \( h = 1 \) ft.

**Hint:** One (but not the only) way to solve this problem is as follows. Use only two (which ones?) of the three boundary conditions (2) and (6) to view this as an *initial value problem* (recall Calculus II). Then use the third boundary condition to find \( L \).

**Appendix A**

Here I present an example of finding the coefficients in a formula similar to Equation (4) of this Project. An analogous derivation is found on pp. 88–89 of the text book.

Recall from Calculus I that the *first* derivative of a function is defined as

\[
y'(a) = \lim_{h \to 0} \frac{1}{h} (y(a + h) - y(a)),
\]

which means that

\[
y'(a) \approx \frac{1}{h} (y(a + h) - y(a)) \quad \text{for sufficiently small } h. \tag{A1}
\]

Note that Equation (A1) is satisfied *exactly* if \( y(x) = \text{const} \) and \( y(x) = x \) (verify this by substituting these two \( y(x) \) into (A1)). If, however, \( y = x^2 \), the right-hand side of Equation (A1) yields \( 2a + h \), which equals \( y'(a) = 2a \) only approximately when \( h \) is small.

Then we ask: Can one modify the expression on the right-hand side of Equation (A1) so that the new expression would *exactly* equal \( y'(x) \) when \( y = x^2 \) (and also when \( y = \text{const} \) and \( y = x \), as before)? The short answer is ‘yes’. The details are provided below.

We look for the required expression in the form given by the right-hand side of Equation (4):

\[
y'(a) \approx A_{-1} y(a - h) + A_0 y(a) + A_1 y(a + h). \tag{A2}
\]

We have required that this equation be satisfied *exactly* rather than approximately whenever \( y = 1 \), \( y = x \), and \( y = x^2 \). Then we simply substitute each of these three functions into Equation (A2):

\[
\text{when } y(x) = 1 : \quad 1' = 0 = A_{-1} \cdot 1 + A_0 \cdot 1 + A_1 \cdot 1 \quad \tag{A3a}
\]

Note that the three “1”s on the r.h.s. occurred because \( y(a - h) = y(a) = y(a + h) = 1 \) when \( y(x) \equiv 1 \).

\[
\text{when } y(x) = x : \quad x' = 1 = A_{-1} \cdot (a - h) + A_0 \cdot (a) + A_1 \cdot (a + h) \quad \tag{A3b}
\]

Similarly to the previous note, the coefficients of \( A_1 \), \( A_0 \), \( A_1 \) came from the fact that \( y(a - h) = a - h \) etc. for \( y(x) = x \).

\[
\text{when } y(x) = x^2 : \quad (x^2)'|_{x=a} = 2a = A_{-1} \cdot (a - h)^2 + A_0 \cdot (a)^2 + A_1 \cdot (a + h)^2 \quad \tag{A3c}
\]

The linear system (A3) for the unknown coefficients \( A_{-1} \), \( A_0 \), \( A_1 \) can be simplified once we notice that we have required it to be true for *any* \( x = a \). In particular, it must be true for \( a = 0 \). Substituting \( a = 0 \) into Equations (A3) and transforming the resulting linear system to the reduced echelon form, we find (verify):

\[
A_{-1} = -1/(2h), \quad A_0 = 0, \quad A_1 = 1/(2h). \quad \tag{A4}
\]
The values for $A_{-1}$, $A_0$, $A_1$ in Equation (4) are obtained similarly, also by requiring that that equation be exact for $y = 1$, $y = x$, and $y = x^2$.

Appendix B

Here I include some hints about MATLAB that may help you in successfully completing this Project.

- Read Appendix A in the text book (you do not need the material from subsections A.4, A.5, A.10, and A.11) and MATLAB Primer by K. Sigmon, posted on the class web site (you do not need the material from pp. 10–14 and 18–21). You may, but do not have to, read other MATLAB resources posted there.

- To open a script file for the first time, start MATLAB, then go to File in the main menu. Select New and then M-file. This opens a window where you can write your script. Save it in some directory on your M: drive. When you want to run your script at any later time, make sure you change the directory (by typing `cd m:\directoryname` in the MATLAB command window) to the directory where your script file is saved.

- The coefficient matrix for solving the linear system for $y_1$, $y_2$, ..., $y_{n-1}$ has the dimension $(n-1) \times (n-1)$. Thus, for $n = 20$, it contains 361 entries. However, only the elements on the main diagonal and the two diagonals closest to it are nonzero (you may easily see this after you complete Step 3 of this assignment). Typing all the 361 entries one by one into the matrix is not a good option. A good option is to use the for-loop of MATLAB (see the Primer, Sec. 6). For example, the loop that defines matrix

$$A = \begin{pmatrix}
5 & 6 & 0 \\
0 & 5 & 6 \\
0 & 0 & 7
\end{pmatrix}$$

is:

```matlab
for i=1:2
    A(i,i)=5;
    A(i,i+1)=6;
end
A(3,3)=7;
```

A few notes are in order.

First, note that the entries which you do not specify are set to zero by default in MATLAB.

Second, note that in the above loop, we defined nonzero entries only of the first two rows of matrix $A$. This is because the indices of such entries in these rows follow a certain pattern. The indices of the nonzero entry in the third row do not follow exactly the same pattern, and hence we had to define that entry outside the loop.

Likewise, in the matrix that you have to set up for this Project, indices of nonzero entries follow the same pattern in most, but not in all, rows. Therefore, first identify those rows with such a pattern and define nonzero entries in these rows within a loop. Then, define nonzero entries whose indices do not follow a pattern, outside the loop.

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4The same letter, M, in the words ‘M-file’ and ‘M: drive’ is a coincidence. There is no relation between MATLAB’s M-files and the CEMS’s M: drive.
Third, in the vector on the right-hand side of your linear system, you should also determine which entries follow a pattern and define them within a loop (or loops). Then, define the entries which do not follow a pattern outside the loop.

Finally, make sure that the matrix you have constructed is indeed the one you intended to construct. You may do so by displaying that matrix for some $n$ less than 20. This, on one hand, will not change the logic of your script and, on the other, will produce a matrix of a smaller dimension, which you should be able to inspect on the screen.

- This comment is about plotting $y(x)$ on the interval $[0, L]$. Note that your solution $y_1, \ldots, y_{n-1}$ does not include the boundary values $y(0)$ and $y(L)$ because they were known, not solved for. Therefore, you need to append these values to your solution vector. See Sec. A.4 in Appendix A of the textbook or Sec. 5 in the Matlab Primer.

If you would like to learn more about numerical methods,

Take the course MATH 337.

Acknowledgement

This project is based on an idea by Professor Scott C. Fulton (Clarkson University) and some material from Chapter 8 of the book “Numerical Analysis: A Practical Approach” by M. Maron and R. Lopez, 3rd Ed., Wadsworth (Belmont, 1991).
Name(s): ____________________________________________

1. (27 points) Approximate formula for $y''(a)$:

2. (15 points) Approximation to the differential equation (1) at the interior point $x_i$:

3. (27 points) Linear system (in matrix form) for the case $n = 5$:

4. (31 points) Solution for the elevation $y_i$ computed for $n = 20$ and rounded to four decimal places:

<table>
<thead>
<tr>
<th>$i$</th>
<th>5</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$, ft</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y_i$, ft</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Attach your code and a Matlab-produced plot of your solution; do not forget to label the axes.

Extra credit (27 points) Attach your work.

$L = \text{ft}$