13.3 CLASSICAL STRAIGHTEDGE AND COMPASS CONSTRUCTIONS

As a simple application of the results we have obtained on algebraic extensions, and in particular on the multiplicativity of extension degrees, we can answer (in the negative) the following geometric problems posed by the Greeks:

I. (Doubling the Cube) Is it possible using only straightedge and compass to construct a cube with precisely twice the volume of a given cube?

II. (Trisecting an Angle) Is it possible using only straightedge and compass to trisect any given angle \( \theta \)?

III. (Squaring the Circle) Is it possible using only straightedge and compass to construct a square whose area is precisely the area of a given circle?

To answer these questions we must translate the construction of lengths by compass and straightedge into algebraic terms. Let 1 denote a fixed given unit distance. Then any distance is determined by its length \( a \in \mathbb{R} \), which allows us to view geometric distances as elements of the real numbers \( \mathbb{R} \). Using the given unit distance 1 to define the scale on the axes, we can then construct the usual Cartesian plane \( \mathbb{R}^2 \) and view all of our constructions as occurring in \( \mathbb{R}^2 \). A point \( (x, y) \in \mathbb{R}^2 \) is then constructible starting with the given distance 1 if and only if its coordinates \( x \) and \( y \) are constructible elements of \( \mathbb{R} \). The problems above then amount to determining whether particular lengths in \( \mathbb{R} \) can be obtained by compass and straightedge constructions from a fixed unit distance. The collection of such real numbers together with their negatives will be called the constructible elements of \( \mathbb{R} \), and we shall not distinguish between the lengths that are constructible and the real numbers that are constructible.

Each straightedge and compass construction consists of a series of operations of the following four types: (1) connecting two given points by a straight line, (2) finding a point of intersection of two straight lines, (3) drawing a circle with given radius and center, and (4) finding the point(s) of intersection of a straight line and a circle or the intersection of two circles.

It is an elementary fact from geometry that if two lengths \( a \) and \( b \) are given one may construct using straightedge and compass the lengths \( a \pm b, ab \) and \( a/b \) (the first two are clear and the latter two are given by the construction of parallel lines (Figure 1)).

![Fig. 1](image-url)

It is also an elementary geometry construction to construct \( \sqrt{a} \) if \( a \) is given: construct the circle with diameter \( 1 + a \) and erect the perpendicular to the diameter as indicated in Figure 2. Then \( \sqrt{a} \) is the length of this perpendicular.
It follows that straightedge and compass constructions give all the algebraic operations of addition, subtraction, multiplication and division (by nonzero elements) in the reals so the collection of constructible elements is a subfield of \( \mathbb{R} \). One can also take square roots of constructible elements. We shall now see that these are essentially the only operations possible.

From the given length 1 it is possible to construct by these operations all the rational numbers \( \mathbb{Q} \). Hence we may construct all of the points \((x, y)\) whose coordinates are rational. We may construct additional elements of \( \mathbb{R} \) by taking square roots, so the collection of elements constructible from 1 of \( \mathbb{R} \) form a field strictly larger than \( \mathbb{Q} \).

The usual formula (“two point form”) for the straight line connecting two points with coordinates in some field \( F \) gives an equation for the line of the form \( ax + by - c = 0 \) with \( a, b, c \in F \). Solving two such equations simultaneously to determine the point of intersection of two such lines gives solutions also in \( F \). It follows that if the coordinates of two points lie in the field \( F \) then straightedge constructions alone will not produce additional points whose coordinates are not also in \( F \).

A compass construction (type (3) or (4) above) defines points obtained by the intersection of a circle with either a straight line or another circle. A circle with center \((h, k)\) and radius \( r \) has equation

\[
(x - h)^2 + (y - k)^2 = r^2
\]

so when we consider the effect of compass constructions on elements of a field \( F \) we are considering simultaneous solutions of such an equation with a linear equation \( ax + by - c = 0 \) where \( a, b, c, h, k, r \in F \), or the simultaneous solutions of two quadratic equations.

In the case of a linear equation and the equation for the circle, solving for \( y \), say, in the linear equation and substituting gives a quadratic equation for \( x \) (and \( y \) is given linearly in terms of \( x \)). Hence the coordinates of the point of intersection are at worst in a quadratic extension of \( F \).

In the case of the intersection of two circles, say

\[
(x - h)^2 + (y - k)^2 = r^2
\]

and

\[
(x - h')^2 + (y - k')^2 = r'^2
\]

subtraction of the second equation from the first shows that we have the same intersection by considering the two equations

\[
(x - h)^2 + (y - k)^2 = r^2
\]

and

\[2(h' - h)x + 2(k' - k)y = r^2 - h^2 - k^2 - r'^2 + h'^2 + k'^2\]
which is the intersection of a circle and a straight line (the straight line connecting the
two points of intersection, in fact) of the type just considered.

It follows that if a collection of constructible elements is given, then one can con-
struct all the elements in the subfield \( F \) of \( \mathbb{R} \) generated by these elements and that any
straightedge and compass operation on elements of \( F \) produces elements in at worst
a quadratic extension of \( F \). Since quadratic extensions have degree 2 and extension
degrees are multiplicative, it follows that if \( \alpha \in \mathbb{R} \) is obtained from elements in a field
\( F \) by a (finite) series of straightedge and compass operations then \( \alpha \) is an element of an
extension \( K \) of \( F \) of degree a power of 2: \( [K : F] = 2^m \) for some \( m \). Since \( [F(\alpha) : F] \)
divides this extension degree, it must also be a power of 2.

**Proposition 23.** If the element \( \alpha \in \mathbb{R} \) is obtained from a field \( F \subset \mathbb{R} \) by a series of
compass and straightedge constructions then \( [F(\alpha) : F] = 2^k \) for some integer \( k \geq 0 \).

**Theorem 24.** None of the classical Greek problems: (I) Doubling the Cube, (II)
Trisecting an Angle, and (III) Squaring the Circle, is possible.

**Proof:** (I) Doubling the cube amounts to constructing \( \sqrt[3]{2} \) in the reals starting with
the unit 1. Since \( [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3 \) is not a power of 2, this is impossible.
(II) If an angle \( \theta \) can be constructed, then determining the point at distance 1 from
the origin and angle \( \theta \) from the positive \( x \) axis in \( \mathbb{R}^2 \) shows that \( \cos \theta \) (the \( x \)-coordinate
of this point) can be constructed (so then \( \sin \theta \) can also be constructed). Conversely if
\( \cos \theta \), then \( \sin \theta \), can be constructed, the point with those coordinates gives the angle \( \theta \).
The problem of trisecting the angle \( \theta \) is then equivalent to the problem: given \( \cos \theta \)
construct \( \cos \theta / 3 \).

To see that this is not always possible (it is certainly occasionally possible, for
example for \( \theta = 180^{\circ} \)), consider \( \theta = 60^{\circ} \). Then \( \cos \theta = \frac{1}{2} \). By the triple angle formula
for cosines:
\[
\cos \theta = 4 \cos^3 \theta / 3 - 3 \cos \theta / 3,
\]
substituting \( \theta = 60^\circ \), we see that \( \beta = \cos 20^\circ \) satisfies the equation
\[
4 \beta^3 - 3 \beta - 1/2 = 0
\]
or \( 8 \beta^3 - 6 \beta - 1 = 0 \). This can be written \( (2 \beta)^3 - 3(2 \beta) - 1 = 0 \). Let \( \alpha = 2 \beta \). Then
\( \alpha \) is a real number between 0 and 2 satisfying the equation
\[
\alpha^3 - 3 \alpha - 1 = 0.
\]
But we considered this equation in the last section and determined \( [\mathbb{Q}(\alpha) : \mathbb{Q}] = 3 \), and
as before we see that \( \alpha \) is not constructible.

(III) Squaring the circle is equivalent to determining whether the real number \( \pi = 3.14159 \ldots \) is constructible. As mentioned previously, it is a difficult problem even
to prove that this number is not rational. It is in fact transcendental (which we shall
assume without proof), so that \( [\mathbb{Q}(\pi) : \mathbb{Q}] \) is not even finite, much less a power of 2,
showing the impossibility of squaring the circle by straightedge and compass.
Remark: The proof above shows that \( \cos 20^\circ \) and \( \sin 20^\circ \) cannot be constructed. The question arises as to which integer angles (measured in degrees) are constructible? The angles \( 1^\circ \) and \( 2^\circ \) are not constructible, since otherwise the addition formulae for sines and cosines would give the constructibility for \( 20^\circ \). On the other hand, elementary geometric constructions (of the regular 5-gon for an angle of \( 72^\circ \) and the equilateral triangle for an angle of \( 60^\circ \)) together with the addition formulae and the half-angle formulae show that \( \cos 3^\circ \) and \( \sin 3^\circ \) are constructible. It follows from this that the trigonometric functions of an integer degree angle are constructible precisely when the angle is a multiple of \( 3^\circ \). Explicitly,

\[
\cos 3^\circ = \frac{1}{8} (\sqrt{3} + 1) \sqrt{5 + \sqrt{5}} + \frac{1}{16} (\sqrt{6} - \sqrt{2})(\sqrt{5} - 1)
\]

\[
\sin 3^\circ = \frac{1}{16} (\sqrt{6} + \sqrt{2})(\sqrt{5} - 1) - \frac{1}{8} (\sqrt{3} - 1) \sqrt{5 + \sqrt{5}},
\]

showing that these are obtained from \( \mathbb{Q} \) by successive extractions of square roots and field operations.

After discussing the cyclotomic fields in Section 14.5 we shall consider another classical geometric question: “which regular \( n \)-gons can be constructed by straightedge and compass?” (cf. Proposition 14.29).

We have been careful here to consider constructions using a \textit{straightedge} rather than a \textit{ruler}, the distinction being that a ruler has marks on it. If one uses a ruler, it is possible to construct many additional algebraic elements. For example, suppose \( \theta \) is a given angle and the unit distance 1 is marked on the ruler. Draw a circle of radius 1 with central angle \( \theta \) as shown in Figure 3 and then slide the ruler until the distance between points \( A \) and \( B \) on the circle is 1. Then some elementary geometry shows that (cf. the exercises) the angle \( \alpha \) indicated is \( \theta / 3 \), i.e., this construction (due to Archimedes) trisects \( \theta \). In particular, the second classical problem in Theorem 24 (Trisecting an Angle) can be solved with \textit{ruler} and compass.

The first of the classical problems in Theorem 24 (Duplication of the Cube), which amounts to the construction of \( \sqrt[3]{2} \), can also be solved with ruler and compass. The following gives a construction for \( k^{1/3} \) for any given positive real \( k \) which is less than 1. This construction was shown to us by J.H. Conway.

Drawing a circle of radius 1 and using the point \( A = (k, 0) \) as center, construct the point \( B = (0, \sqrt{1 - k^2}) \). Dividing this distance by 3, construct the point \( (0, -\frac{1}{3} \sqrt{1 - k^2}) \) and draw the line connecting this point with \( A \). Slide the ruler with
marked unit length 1 so that it passes through the point $B$ and so that the distance from the intersection point $C$ to the intersection point $D$ with the $x$-axis is of length 1, as indicated in Figure 4.

Then the distance between $A$ and $D$ is $2k^{1/3}$ and the distance between $B$ and $C$ is $2k^{2/3}$ (cf. the exercises).

**EXERCISES**

1. Prove that it is impossible to construct the regular 9-gon.

2. Prove that Archimedes’ construction actually trisects the angle $\theta$. [Note the isosceles triangles in Figure 5 to prove that $\beta = \gamma = 2\alpha$.]

3. Prove that Conway’s construction indicated in the text actually constructs $2k^{1/3}$ and $2k^{2/3}$. [One method: let $(x, y)$ be the coordinates of the point $C$, $a$ the distance from $B$ to $C$ and $b$ the distance from $A$ to $D$; use similar triangles to prove (a) $\frac{y}{1} = \frac{\sqrt{1-k^2}}{1+a}$, (b) $\frac{x}{a} = \frac{b+k}{1+a}$, (c) $\frac{y}{x-k} = \frac{\sqrt{1-k^2}}{3k}$, and also show that (d) $(1-k^2)+(b+k)^2 = (1+a)^2$; solve these equations for $a$ and $b$.]

4. The construction of the regular 7-gon amounts to the constructibility of $\cos(2\pi/7)$. We shall see later (Section 14.5 and Exercise 2 of Section 14.7) that $\alpha = 2 \cos(2\pi/7)$ satisfies the equation $x^3 + x^2 - 2x - 1 = 0$. Use this to prove that the regular 7-gon is not constructible by straightedge and compass.

5. Use the fact that $\alpha = 2 \cos(2\pi/5)$ satisfies the equation $x^2 + x - 1 = 0$ to conclude that the regular 5-gon is constructible by straightedge and compass.