One of the main goals of this book is to help you develop a solid and practical understanding of bifurcations. This chapter introduces the simplest examples: bifurcations of fixed points for flows on the line. We'll use these bifurcations to model such dramatic phenomena as the onset of coherent radiation in a laser and the outbreak of an insect population. (In later chapters, when we step up to two- and three-dimensional phase spaces, we'll explore additional types of bifurcations and their scientific applications.)

We begin with the most fundamental bifurcation of all.

### 3.1 Saddle-Node Bifurcation

The saddle-node bifurcation is the basic mechanism by which fixed points are created and destroyed. As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

The prototypical example of a saddle-node bifurcation is given by the first-order system

$$\dot{x} = r + x^2$$  \hspace{1cm} (1)

where $r$ is a parameter, which may be positive, negative, or zero. When $r$ is negative, there are two fixed points, one stable and one unstable (Figure 3.1.1a).

![Figure 3.1.1](image)

As $r$ approaches 0 from below, the parabola moves up and the two fixed points move toward each other. When $r = 0$, the fixed points coalesce into a half-stable fixed point at $x^* = 0$ (Figure 3.1.1b). This type of fixed point is extremely delicate—it vanishes as soon as $r > 0$, and now there are no fixed points at all (Figure 3.1.1c).

In this example, we say that a bifurcation occurred at $r = 0$, since the vector fields for $r < 0$ and $r > 0$ are qualitatively different.

**Graphical Conventions**

There are several other ways to depict a saddle-node bifurcation. We can show a stack of vector fields for discrete values of $r$ (Figure 3.1.2).
This representation emphasizes the dependence of the fixed points on \( r \). In the limit of a continuous stack of vector fields, we have a picture like Figure 3.1.3. The curve shown is \( r = -x^2 \), i.e., \( \dot{x} = 0 \), which gives the fixed points for different \( r \). To distinguish between stable and unstable fixed points, we use a solid line for stable points and a broken line for unstable ones.

However, the most common way to depict the bifurcation is to invert the axes of Figure 3.1.3. The rationale is that \( r \) plays the role of an independent variable, and so should be plotted horizontally.

(\section{3.1.3}) The drawback is that now the \( x \)-axis has to be plotted vertically, which looks strange at first. Arrows are sometimes included in the picture, but not always. This picture is called the bifurcation diagram for the saddle-node bifurcation.

**Terminology**

Bifurcation theory is rife with conflicting terminology. The subject really hasn't settled down yet, and different people use different words for the same thing. For example, the saddle-node bifurcation is sometimes called a fold bifurcation (because the curve in Figure 3.1.4 has a fold in it) or a turning-point bifurcation (because the point \((x,r) = (0,0)\) is a "turning point.") Admittedly, the term saddle-node doesn't make much sense for vector fields on the line. The name derives from a completely analogous bifurcation seen in a higher-dimensional context, such as vector fields on the plane, where fixed points known as saddles and nodes can collide and annihilate (see Section 8.1).

The prize for most inventive terminology must go to Abraham and Shaw (1988), who write of a blue sky bifurcation. This term comes from viewing a saddle-node bifurcation in the other direction: a pair of fixed points appears "out of the clear blue sky" as a parameter is varied. For example, the vector field

\[
\dot{x} = r - x^2
\]

has no fixed points for \( r < 0 \), but then one materializes when \( r = 0 \) and splits into two when \( r > 0 \) (Figure 3.1.5). Incidentally, this example also explains why we use the word "bifurcation": it means "splitting into two branches."

\section{3.1.1:}

**Example 3.1.1:**

Give a linear stability analysis of the fixed points in Figure 3.1.5.

**Solution:** The fixed points for \( \dot{x} = f(x) = r - x^2 \) are given by \( x^* = \pm \sqrt{r} \). There are two fixed points for \( r \geq 0 \), and none for \( r < 0 \). To determine linear stability, we compute \( f'(x^*) = -2x^* \). Thus \( x^* = + \sqrt{r} \) is stable, since \( f'(x^*) < 0 \). Similarly \( x^* = - \sqrt{r} \) is unstable. At the bifurcation point \( r = 0 \), we find \( f'(x^*) = 0 \); the linearization vanishes when the fixed points coalesce. 

\section{3.1.2:}

**Example 3.1.2:**

Show that the first-order system \( \dot{x} = r - x - e^{-x} \) undergoes a saddle-node bifurcation as \( r \) is varied, and find the value of \( r \) at the bifurcation point.

**Solution:** The fixed points satisfy \( f(x) = r - x - e^{-x} = 0 \). But now we run into difficulty—in contrast to Example 3.1.1, we can't find the fixed points explicitly as a function of \( r \). Instead we adopt a geometric approach. One method could be to graph the function \( f(x) = r - x - e^{-x} \) for different values of \( r \), look for its roots \( x^* \), and then sketch the vector field on the \( x \)-axis. This method is
fine, but there’s an easier way. The point is that the two functions $r-x$ and $e^{-x}$ have much more familiar graphs than their difference $r-x-e^{-x}$. So we plot $r-x$ and $e^{-x}$ on the same picture (Figure 3.1.6a). Where the line $r-x$ intersects the curve $e^{-x}$, we have $r-x = e^{-x}$ and so $f(x) = 0$. Thus, intersections of the line and the curve correspond to fixed points for the system. This picture also allows us to read off the direction of flow on the $x$-axis: the flow is to the right where the line lies above the curve, since $r-x > e^{-x}$ and therefore $x > 0$. Hence, the fixed point on the right is stable, and the one on the left is unstable.

Now imagine we start decreasing the parameter $r$. The line $r-x$ slides down and the fixed points approach each other. At some critical value $r = r_c$, the line becomes tangent to the curve and the fixed points coalesce in a saddle-node bifurcation (Figure 3.1.6b). For $r$ below this critical value, the line lies below the curve and there are no fixed points (Figure 3.1.6c).

For instance, consider Example 3.1.2 near the bifurcation at $x = 0$ and $r = 1$. Using the Taylor expansion for $e^{-x}$ about $x = 0$, we find

$$\dot{x} = r-x - e^{-x}$$
$$= r-x - \left[1 - x + \frac{x^2}{2!} + \cdots\right]$$
$$= (r-1) - \frac{x^2}{2} + \cdots$$

to leading order in $x$. This has the same algebraic form as $\dot{x} = r-x^2$, and can be made to agree exactly by appropriate rescalings of $x$ and $r$.

It’s easy to understand why saddle-node bifurcations typically have this algebraic form. We just ask ourselves: how can two fixed points of $\dot{x} = f(x)$ collide and disappear as a parameter $r$ is varied? Graphically, fixed points occur where the graph of $f(x)$ intersects the $x$-axis. For a saddle-node bifurcation to be possible, we need two nearby roots of $f(x)$; this means $f(x)$ must look locally “bowl-shaped” or parabolic (Figure 3.1.7).

Normal Forms

In a certain sense, the examples $\dot{x} = r-x^2$ or $\dot{x} = r+x^2$ are representative of all saddle-node bifurcations; that’s why we called them “prototypical.” The idea is that, close to a saddle-node bifurcation, the dynamics typically look like $\dot{x} = r-x^2$ or $\dot{x} = r+x^2$.

Now we use a microscope to zoom in on the behavior near the bifurcation. As $r$ varies, we see a parabola intersecting the $x$-axis, then becoming tangent to it, and then failing to intersect. This is exactly the scenario in the prototypical Figure 3.1.1. Here’s a more algebraic version of the same argument. We regard $f$ as a function of both $x$ and $r$, and examine the behavior of $\dot{x} = f(x,r)$ near the bifurcation at $x = x^*$ and $r = r_c$. Taylor’s expansion yields

$$\dot{x} = f(x,r)$$
$$= f(x^*,r_c) + (x-x^*) \frac{\partial f}{\partial x} \bigg|_{(x^*,r_c)} + (r-r_c) \frac{\partial f}{\partial r} \bigg|_{(x^*,r_c)} + \frac{1}{2} (x-x^*)^2 \frac{\partial^2 f}{\partial x^2} \bigg|_{(x^*,r_c)} + \cdots$$

3.1 SADDLE-NODE BIFURCATION
where we have neglected quadratic terms in \((r-r_\ast)\) and cubic terms in \((x-x^\ast)\). Two of the terms in this equation vanish: \(f(x_\ast,r) = 0\) since \(x^\ast\) is a fixed point, and \(\partial f/\partial x_{r,x_\ast} = 0\) by the tangency condition of a saddle-node bifurcation. Thus

\[
\dot{x} = a(r-r_\ast) + b(x-x^\ast)^2 + \ldots \tag{3}
\]

where \(a = \partial f/\partial x_{r,x_\ast}\), and \(b = \frac{1}{2} \partial^2 f/\partial x^2_{r,x_\ast}\). Equation (3) agrees with the form of our prototypical examples. (We are assuming \(a, b \neq 0\), which is the typical case; for instance, it would be a very special situation if the second derivative \(\partial^2 f/\partial x^2\) also happened to vanish at the fixed point.)

What we have been calling prototypical examples are more conventionally known as normal forms for the saddle-node bifurcation. There is much, much more to normal forms than we have indicated here. We will be seeing their importance throughout this book. For a more detailed and precise discussion, see Guckenheimer and Holmes (1983) or Wiggins (1990).

### 3.2 Transcritical Bifurcation

There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed. For example, in the logistic equation and other simple models for the growth of a single species, there is a fixed point at zero population, regardless of the value of the growth rate. However, such a fixed point may change its stability as the parameter is varied. The transcritical bifurcation is the standard mechanism for such changes in stability.

The normal form for a transcritical bifurcation is

\[
\dot{x} = x - x^2. \tag{1}
\]

This looks like the logistic equation of Section 2.3, but now we allow \(x\) and \(r\) to be either positive or negative.

Figure 3.2.1 shows the vector field as \(r\) varies. Note that there is a fixed point at \(x^\ast = 0\) for all values of \(r\).

For \(r < 0\), there is an unstable fixed point at \(x^\ast = r\) and a stable fixed point at \(x^\ast = 0\). As \(r\) increases, the unstable fixed point approaches the origin, and coalesces with it when \(r = 0\). Finally, when \(r > 0\), the origin has become unstable, and \(x^\ast = r\) is now stable. Some people say that an exchange of stabilities has taken place between the two fixed points.

Please note the important difference between the saddle-node and transcritical bifurcations: in the transcritical case, the two fixed points don't disappear after the bifurcation—instead they just switch their stability.

Figure 3.2.2 shows the bifurcation diagram for the transcritical bifurcation. As in Figure 3.1.4, the parameter \(r\) is regarded as the independent variable, and the fixed points \(x^\ast = 0\) and \(x^\ast = r\) are shown as dependent variables.

**Example 3.2.1:**

Show that the first-order system \(\dot{x} = x(1-x^2) - a(1-e^{bx})\) undergoes a transcritical bifurcation at \(x = 0\) when the parameters \(a, b\) satisfy a certain equation, to be determined. (This equation defines a bifurcation curve in the \((a,b)\) parameter space.) Then find an approximate formula for the fixed point that bifurcates from \(x = 0\), assuming that the parameters are close to the bifurcation curve.

**Solution:** Note that \(x = 0\) is a fixed point for all \((a,b)\). This makes it plausible that the fixed point will bifurcate transcritically, if it bifurcates at all. For small \(x\), we find

\[
1-e^{ bx} = 1 - [1 - bx + \frac{1}{2}b^2x^2 + O(x^3)]
\]

\[
= bx - \frac{1}{2}b^2x^2 + O(x^3)
\]

and so

\[
\dot{x} = x - a(bx - \frac{1}{2}b^2x^2 + O(x^3))
\]

\[
= (1-ab)x + (\frac{1}{2}ab^2)x^2 + O(x^3).
\]

---

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**3.2 TRANSCRITICAL BIFURCATION** **51**
Hence a transcritical bifurcation occurs when \( ab = 1 \); this is the equation for the bifurcation curve. The nonzero fixed point is given by the solution of \( 1 - ab + (ab^2)x = 0 \), i.e.,
\[
x^* = \frac{2(ab-1)}{ab^2}.
\]
This formula is approximately correct only if \( x^* \) is small, since our series expansions are based on the assumption of small \( x \). Thus the formula holds only when \( ab \) is close to 1, which means that the parameters must be close to the bifurcation curve.

**Example 3.2.2:**

Analyze the dynamics of \( \dot{x} = r\ln x + x - 1 \) near \( x = 1 \), and show that the system undergoes a transcritical bifurcation at a certain value of \( r \). Then find new variables \( X \) and \( R \) such that the system reduces to the approximate normal form \( \dot{X} = RX - X^2 \) near the bifurcation.

**Solution:** First note that \( x = 1 \) is a fixed point for all values of \( r \). Since we are interested in the dynamics near this fixed point, we introduce a new variable \( u = x - 1 \), where \( u \) is small. Then
\[
\dot{u} = \dot{x}
= r\ln(1+u) + u
= r[u - \frac{1}{2}u^2 + O(u^3)] + u
= (r + 1)u - \frac{1}{2}ru^2 + O(u^3).
\]
Hence a transcritical bifurcation occurs at \( r_\ast = -1 \).

To put this equation into normal form, we first need to get rid of the coefficient of \( u^2 \). Let \( u = av \), where \( a \) will be chosen later. Then the equation for \( v \) is
\[
\dot{v} = (r + 1)v - \frac{1}{2}rv^2 + O(v^3).
\]
So if we choose \( a = 2/r \), the equation becomes
\[
\dot{v} = (r + 1)v - v^2 + O(v^3).
\]
Now if we let \( R = r + 1 \) and \( X = v \), we have achieved the approximate normal form \( \dot{X} = RX - X^2 \), where cubic terms of order \( O(X^3) \) have been neglected. In terms of the original variables, \( X = v = u/a = \frac{1}{2}r(x-1) \).

To be a bit more accurate, the theory of normal forms assures us that we can find a change of variables such that the system becomes \( X = RX - X^2 \), with strict, rather than approximate, equality. Our solution above gives an approximation to the necessary change of variables. If we wanted a better approximation, we would retain the cubic terms in the series expansions (and perhaps even higher-order terms if we're really feeling heroic) and we would have to do a more elaborate calculation to eliminate these higher-order terms. See Exercises 3.2.6 and 3.2.7 for a taste of such calculations, or see the books of Guckenheimer and Holmes (1983), Wiggins (1990), or Manneville (1990).

### 3.3 Laser Threshold

Now it's time to apply our mathematics to a scientific example. We analyze an extremely simplified model for a laser, following the treatment given by Haken (1983).

**Physical Background**

We are going to consider a particular type of laser known as a solid-state laser, which consists of a collection of special "laser-active" atoms embedded in a solid-state matrix, bounded by partially reflecting mirrors at either end. An external energy source is used to excite or "pump" the atoms out of their ground states (Figure 3.3.1).

![Figure 3.3.1](image)

Each atom can be thought of as a little antenna radiating energy. When the pumping is relatively weak, the laser acts just like an ordinary lamp: the excited atoms oscillate independently of one another and emit randomly phased light waves.

Now suppose we increase the strength of the pumping. At first nothing different happens, but then suddenly, when the pump strength exceeds a certain threshold, the atoms begin to oscillate in phase—the lamp has turned into a laser. Now the trillions of little antennas act like one giant antenna and produce a beam of radiation that is much more coherent and intense than that produced below the laser threshold.

This sudden onset of coherence is amazing, considering that the atoms are being excited completely at random by the pump! Hence the process is self-organizing: the coherence develops because of a cooperative interaction among the atoms themselves.

**Model**

A proper explanation of the laser phenomenon would require us to delve into quantum mechanics. See Milonni and Eberly (1988) for an intuitive discussion.
Instead we consider a simplified model of the essential physics (Haken 1983, p. 127). The dynamical variable is the number of photons \( n(t) \) in the laser field. Its rate of change is given by

\[
\dot{n} = \text{gain} - \text{loss} = G n N - k n.
\]

The gain term comes from the process of stimulated emission, in which photons stimulate excited atoms to emit additional photons. Because this process occurs via random encounters between photons and excited atoms, it occurs at a rate proportional to \( n \) and to the number of excited atoms, denoted by \( N(t) \). The parameter \( G > 0 \) is known as the gain coefficient. The loss term models the escape of photons through the endfaces of the laser. The parameter \( k > 0 \) is a rate constant; its reciprocal \( \tau = 1/k \) represents the typical lifetime of a photon in the laser.

Now comes the key physical idea: after an excited atom emits a photon, it drops down to a lower energy level and is no longer excited. Thus \( N \) decreases by the emission of photons. To capture this effect, we need to write an equation relating \( N \) to \( n \). Suppose that in the absence of laser action, the pump keeps the number of excited atoms fixed at \( N_0 \). Then the actual number of excited atoms will be reduced by the laser process. Specifically, we assume

\[
N(t) = N_0 - \alpha n,
\]

where \( \alpha > 0 \) is the rate at which atoms drop back to their ground states. Then

\[
\dot{n} = G n (N_0 - \alpha n) - k n = (G N_0 - k n - (\alpha G) n^2).
\]

We’re finally on familiar ground—this is a first-order system for \( n(t) \). Figure 3.3.2 shows the corresponding vector field for different values of the pump strength \( N_0 \). Note that only positive values of \( n \) are physically meaningful.

When \( N_0 < k/G \), the fixed point at \( n^* = 0 \) is stable. This means that there is no stimulated emission and the laser acts like a lamp. As the pump strength \( N_0 \) is increased, the system undergoes a transcritical bifurcation when \( N_0 = k/G \). For \( N_0 > k/G \), the origin loses stability and a stable fixed point appears at \( n^* = (GN_0 - k)/(\alpha G) > 0 \), corresponding to spontaneous laser action. Thus \( N_0 = k/G \) can be interpreted as the laser threshold in this model. Figure 3.3.3 summarizes our results.

Although this model correctly predicts the existence of a threshold, it ignores the dynamics of the excited atoms, the existence of spontaneous emission, and several other complications. See Exercises 3.3.1 and 3.3.2 for improved models.

### 3.4 Pitchfork Bifurcation

We turn now to a third kind of bifurcation, the so-called pitchfork bifurcation. This bifurcation is common in physical problems that have a symmetry. For example, many problems have a spatial symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs. In the buckling example of Figure 3.0.1, the beam is stable in the vertical position if the load is small. In this case there is a stable fixed point corresponding to zero deflection. But if the load exceeds the buckling threshold, the beam may buckle to either the left or the right. The vertical position has gone unstable, and two new symmetrical fixed points, corresponding to left- and right-buckled configurations, have been born.

There are two very different types of pitchfork bifurcation. The simpler type is called supercritical, and will be discussed first.

**Supercritical Pitchfork Bifurcation**

The normal form of the supercritical pitchfork bifurcation is

\[
\dot{x} = rx - x^3. \tag{1}
\]
Note that this equation is **invariant** under the change of variables $x \rightarrow -x$. That is, if we replace $x$ by $-x$ and then cancel the resulting minus signs on both sides of the equation, we get (1) back again. This invariance is the mathematical expression of the left-right symmetry mentioned earlier. (More technically, one says that the vector field is **equivariant**, but we'll use the more familiar language.)

Figure 3.4.1 shows the vector field for different values of $r$.

When $r < 0$, the origin is the only fixed point, and it is stable. When $r = 0$, the origin is still stable, but much more weakly so, since the linearization vanishes. Now solutions no longer decay exponentially fast—instead the decay is a much slower algebraic function of time (recall Exercise 2.4.9). This lethargic decay is called **critical slowing down** in the physics literature. Finally, when $r > 0$, the origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x^* = \pm \sqrt{r}$.

The reason for the term ”pitchfork” becomes clear when we plot the bifurcation diagram (Figure 3.4.2). Actually, pitchfork bifurcation might be a better word!

---

**Example 3.4.1:**

Equations similar to $\dot{x} = -x + \beta \tanh x$ arise in statistical mechanical models of magnets and neural networks (see Exercise 3.6.7 and Palmer 1989). Show that this equation undergoes a supercritical pitchfork bifurcation as $\beta$ is varied. Then give a **numerically accurate** plot of the fixed points for each $\beta$.

**Solution:** We use the strategy of Example 3.1.2 to find the fixed points. The graphs of $y = x$ and $y = \beta \tanh x$ are shown in Figure 3.4.3; their intersections correspond to fixed points. The key thing to realize is that as $\beta$ increases, the tanh curve becomes steeper at the origin (its slope there is $\beta$). Hence for $\beta < 1$ the origin is the only fixed point. A pitchfork bifurcation occurs at $\beta = 1$, $x^* = 0$, when the tanh curve develops a slope of 1 at the origin. Finally, when $\beta > 1$, two new stable fixed points appear, and the origin becomes unstable.

---

**Figure 3.4.3**

Now we want to compute the fixed points $x^*$ for each $\beta$. Of course, one fixed point always occurs at $x^* = 0$; we are looking for the other, nontrivial fixed points. One approach is to solve the equation $x^* = \beta \tanh x^*$ numerically, using the Newton–Raphson method or some other root-finding scheme. (See Press et al. (1986) for a friendly and informative discussion of numerical methods.)

But there’s an easier way, which comes from changing our point of view. Instead of studying the dependence of $x^*$ on $\beta$, we think of $x^*$ as the independent variable, and then compute $\beta = x^*/\tanh x^*$. This gives us a table of pairs $(x^*, \beta)$. For each pair, we plot $\beta$ horizontally and $x^*$ vertically. This yields the bifurcation diagram (Figure 3.4.4).

---

**Figure 3.4.4**

---

**3.4 Pitchfork Bifurcation**
The shortcut used here exploits the fact that \( f(x, \beta) = -x + \beta \tanh x \) depends more simply on \( \beta \) than on \( x \). This is frequently the case in bifurcation problems—the dependence on the control parameter is usually simpler than the dependence on \( x \).

**Example 3.4.2:**

Plot the potential \( V(x) \) for the system \( \dot{x} = rx - x^3 \), for the cases \( r < 0, \ r = 0, \) and \( r > 0 \).

**Solution:** Recall from Section 2.7 that the potential for \( \dot{x} = f(x) \) is defined by \( f(x) = -\frac{dV}{dx} \). Hence we need to solve \( -\frac{dV}{dx} = rx - x^3 \). Integration yields \( V(x) = -\frac{r}{2} x^2 + \frac{1}{4} x^4 \), where we neglect the arbitrary constant of integration. The corresponding graphs are shown in Figure 3.4.5.

![Figure 3.4.5](image)

When \( r < 0 \), there is a quadratic minimum at the origin. At the bifurcation value \( r = 0 \), the minimum becomes a much flatter quartic. For \( r > 0 \), a local maximum appears at the origin, and a symmetric pair of minima occur to either side of it.

**Subcritical Pitchfork Bifurcation**

In the supercritical case \( \dot{x} = rx - x^3 \) discussed above, the cubic term is stabilizing: it acts as a restoring force that pulls \( x(t) \) back toward \( x = 0 \). If instead the cubic term were destabilizing, as in

\[
\dot{x} = rx + x^3,
\]

then we'd have a subcritical pitchfork bifurcation. Figure 3.4.6 shows the bifurcation diagram.

![Figure 3.4.6](image)

Compared to Figure 3.4.2, the pitchfork is inverted. The nonzero fixed points \( x = \pm \sqrt{r} \) are unstable, and exist only below the bifurcation \( (r < 0) \), which motivates the term “subcritical.” More importantly, the origin is stable for \( r < 0 \) and unstable for \( r > 0 \), as in the supercritical case, but now the instability for \( r > 0 \) is not opposed by the cubic term—in fact the cubic term lends a helping hand in driving the trajectories out to infinity! This effect leads to blow-up: one can show that \( x(t) \to \pm \infty \) in finite time, starting from any initial condition \( x_0 \neq 0 \) (Exercise 2.5.3).

In real physical systems, such an explosive instability is usually opposed by the stabilizing influence of higher-order terms. Assuming that the system is still symmetric under \( x \to -x \), the first stabilizing term must be \( x^3 \). Thus the canonical example of a system with a subcritical pitchfork bifurcation is

\[
\dot{x} = rx + x^3 - x^5.
\]  

There's no loss in generality in assuming that the coefficients of \( x^3 \) and \( x^5 \) are 1 (Exercise 3.5.8).

The detailed analysis of (3) is left to you (Exercises 3.4.14 and 3.4.15). But we will summarize the main results here. Figure 3.4.7 shows the bifurcation diagram for (3).

![Figure 3.4.7](image)

For small \( x \), the picture looks just like Figure 3.4.6: the origin is locally stable for \( r < 0 \), and two backward-bending branches of unstable fixed points bifurcate from the origin when \( r = 0 \). The new feature, due to the \( x^5 \) term, is that the unstable branches turn around and become stable at \( r = r_c \), where \( r_c < 0 \). These stable large-amplitude branches exist for all \( r > r_c \).
There are several things to note about Figure 3.4.7:

1. In the range \( r < r < 0 \), two qualitatively different stable states coexist, namely the origin and the large-amplitude fixed points. The initial condition \( x_0 \) determines which fixed point is approached as \( t \to \infty \). One consequence is that the origin is stable to small perturbations, but not to large ones—in this sense the origin is \textit{locally} stable, but not \textit{globally} stable.

2. The existence of different stable states allows for the possibility of \textit{jumps} and \textit{hysteresis} as \( r \) is varied. Suppose we start the system in the state \( x^* = 0 \), and then slowly increase the parameter \( r \) (indicated by an arrow along the \( r \)-axis of Figure 3.4.8).

![Figure 3.4.8](image)

Then the state remains at the origin until \( r = 0 \), when the origin loses stability. Now the slightest nudge will cause the state to \textit{jump} to one of the large-amplitude branches. With further increases of \( r \), the state moves out along the large-amplitude branch. If \( r \) is now decreased, the state remains on the large-amplitude branch, even when \( r \) is decreased below 0! We have to lower \( r \) even further (down past \( r_f \)) to get the state to jump back to the origin. This lack of reversibility as a parameter is varied is called \textit{hysteresis}.

3. The bifurcation at \( r_f \) is a saddle-node bifurcation, in which stable and unstable fixed points are born “out of the clear blue sky” as \( r \) is increased (see Section 3.1).

**Terminology**

As usual in bifurcation theory, there are several other names for the bifurcations discussed here. The supercritical pitchfork is sometimes called a forward bifurcation, and is closely related to a continuous or second-order phase transition in statistical mechanics. The subcritical bifurcation is sometimes called an inverted or backward bifurcation, and is related to discontinuous or first-order phase transitions. In the engineering literature, the supercritical bifurcation is sometimes called soft or safe, because the nonzero fixed points are born at small amplitude; in contrast, the subcritical bifurcation is hard or dangerous, because of the jump from zero to large amplitude.

### 3.5 Overdamped Bead on a Rotating Hoop

In this section we analyze a classic problem from first-year physics, the bead on a rotating hoop. This problem provides an example of a bifurcation in a mechanical system. It also illustrates the subtleties involved in replacing Newton’s law, which is a second-order equation, by a simpler first-order equation.

The mechanical system is shown in Figure 3.5.1. A bead of mass \( m \) slides along a wire hoop of radius \( r \). The hoop is constrained to rotate at a constant angular velocity \( \omega \) about its vertical axis. The problem is to analyze the motion of the bead, given that it is acted on by both gravitational and centrifugal forces. This is the usual statement of the problem, but now we want to add a new twist: suppose that there’s also a frictional force on the bead that opposes its motion. To be specific, imagine that the whole system is immersed in a vat of molasses or some other very viscous fluid, and that the friction is due to viscous damping.

Let \( \phi \) be the angle between the bead and the downward vertical direction. By convention, we restrict \( \phi \) to the range \(-\pi < \phi \leq \pi \), so there’s only one angle for each point on the hoop. Also, let \( p = r \sin \phi \) denote the distance of the bead from the vertical axis. Then the coordinates are as shown in Figure 3.5.2.

![Figure 3.5.1](image)

Next we write Newton’s law for the bead. There’s a downward gravitational force \( mg \), a sideways centrifugal force \( m r \omega^2 \), and a tangential damping force \( b \phi \). (The constants \( g \) and \( b \) are taken to be positive; negative signs will be added later as needed.) The hoop is assumed to be rigid, so we only have to resolve the forces along the tangential direction, as shown in Figure 3.5.3. After substituting \( \rho = r \sin \phi \) in the centrifugal term, and recalling that the tangential acceleration is \( \rho \ddot{\phi} \), we obtain the governing equation

\[
\rho \ddot{\phi} = -b \phi - mg \sin \phi + mr^2 \phi \sin \phi \cos \phi.
\]
This is a second-order differential equation, since the second derivative $\phi$ is the highest one that appears. We are not yet equipped to analyze second-order equations, so we would like to find some conditions under which we can safely neglect the $mr\phi$ term. Then (1) reduces to a first-order equation, and we can apply our machinery to it.

Of course, this is a dicey business: we can’t just neglect terms because we feel like it! But we will for now, and then at the end of this section we'll try to find a regime where our approximation is valid.

**Analysis of the First-Order System**

Our concern now is with the first-order system

$$b \phi = -mg \sin \phi + m r \omega^2 \sin \phi \cos \phi$$

$$= mg \sin \phi \left( \frac{r \omega^2}{g} \cos \phi - 1 \right).$$

The fixed points of (2) correspond to equilibrium positions for the bead. What's your intuition about where such equilibria can occur? We would expect the bead to remain at rest if placed at the top or the bottom of the hoop. Can other fixed points occur? And what about stability? Is the bottom always stable?

Equation (2) shows that there are always fixed points where $\sin \phi = 0$, namely $\phi^* = 0$ (the bottom of the hoop) and $\phi^* = \pi$ (the top). The more interesting result is that there are two additional fixed points if

$$\frac{r \omega^2}{g} > 1,$$

that is, *if the hoop is spinning fast enough*. These fixed points satisfy $\phi^* = \pm \cos^{-1} \left( \frac{g}{r \omega^2} \right)$. To visualize them, we introduce a parameter

$$\gamma = \frac{r \omega^2}{g}$$

and solve $\cos \phi^* = 1/\gamma$ graphically. We plot $\cos \phi$ vs. $\phi$, and look for intersections with the constant function $1/\gamma$, shown as a horizontal line in Figure 3.5.4. For $\gamma < 1$ there are no intersections, whereas for $\gamma > 1$ there is a symmetrical pair of inter-

**Figure 3.5.4**

$\gamma < 1$ \hspace{1cm} $\gamma > 1$

To summarize our results so far, let's plot all the fixed points as a function of the parameter $\gamma$ (Figure 3.5.6). As usual, solid lines denote stable fixed points and broken lines denote unstable fixed points.

**Figure 3.5.5**

**Figure 3.5.6**

$\gamma$