Lecture 20. Bifurcations of the phase portrait

Here we will consider how qualitative changes to the phase portrait can occur when a parameter, say $\mu$, in equations

$$\begin{align*}
\dot{x} &= f(x, y, \mu) \\
\dot{y} &= g(x, y, \mu),
\end{align*}$$

(1)

is changed. The portrait can change in two ways:

- the number of equilibria is changed,

- or

- the number of equilibria remains the same, but their types change.

The number of equilibria can change in both conservative and nonconservative systems, but their types can change only in nonconservative ones.

1. Bifurcations where the number of equilibria changes in nonconservative systems.

The bifurcations are just 2D generalizations of the 1D saddle-node and pitchfork bifurcations.

A

Saddle-node bifurcation

The simplest form of the corresponding equations, called the normal form, is:
\[ \begin{align*}
x' &= \mu + x^2 \\
y' &= -y \tag{2}
\end{align*} \]

Let us draw its nullclines.

\[
\begin{align*}
x = 0 &\Rightarrow f(x, y) = 0 \Rightarrow \mu + x^2 = 0 \\
y = 0 &\Rightarrow g(x, y) = 0 \Rightarrow y = 0.
\end{align*}
\]

The equilibria occur where the nullclines intersect. The number of such intersections changes when \( \mu \) crosses the zero value.

\[ \mu > 0 \] There is no \((x=0)\)-nullcline, hence no equilibria.

\[ \mu < 0 \] A pair saddle-node "is born" out of nowhere.
Question: Equations (2) have a very special form, so how can they be applicable in general?

Answer: A saddle-node bifurcation occurs whenever the nullclines intersect like this:

\[ \mu > 0 \]

The nullclines do not intersect, so there are no equilibria.

\[ \mu < 0 \]

So, locally, i.e. near the value \( \mu_c \) where the bifurcation occurs (\( \mu = 0 \) above), the vicinity of the intersections of the two nullclines looks like this:

\[ (\mu - \mu_c) = \text{one sign} \]

No second nullcline anywhere near nullcline 2.

\[ (\mu - \mu_c) = \text{opposite sign} \]

Parts of nullcline 2 intersecting nullcline 1.
Ex. 1 Find μc and the locations of the equilibria when \( μ = μc \) for the system

\[
\dot{x} = μ + 2x - x^2 - y
\]
\[
\dot{y} = -y
\]

Solution:

\( (\dot{x} = 0) \)-nullcline \( \Rightarrow μ + 2x - x^2 - y = 0 \)

\( (\dot{y} = 0) \)-nullcline \( \Rightarrow y = 0 \)

The \( (\dot{x} = 0) \)-nullcline can be put in normal form:

\[
y = μ + 2x - x^2 = μ - (x^2 - 2x) = μ - (x^2 - 2x + 1 - 1) = μ - (x - 1)^2 - 1 = (μ + 1) - (x - 1)^2.
\]

Thus, \( μc = -1 \), and the phase portrait is:

\[
\begin{align*}
\mu + 1 < 0 & \quad \text{y} & \quad \mu + 1 > 0 \\
(\dot{y} = 0) \text{-nullcline} & \quad \text{y} & \quad \text{y}
\end{align*}
\]

saddle at \((1 - \sqrt{μ + 1}, 0)\)
stable node at \((1 + \sqrt{μ + 1}, 0)\).

A slightly more complicated, but more "physical," example is found in Ex. 8.1.1, of Strogatz (p. 243).
Another bifurcation that we considered in Lecture 16 where the number of equilibria changed, was the pitchfork bifurcation.

**Supercritical pitchfork bifurcation in 2D**

Normal form:

\[ \dot{x} = \mu x - x^3 \]
\[ \dot{y} = -y \]  \hspace{1cm} (3)

For \( \mu < 0 \):

Note that here, a pair saddle node is born in addition to the already existing stable node.

**Subcritical pitchfork bifurcation in 2D**

Normal form:

\[ \dot{x} = \mu x + x^3 \], \[ \dot{y} = -y \]  \hspace{1cm} (4)

For \( \mu > 0 \):
A more general picture for these bifurcations is:

**Supercritical pitchfork**

\[ (\mu-M_c) < 0 \quad (\mu-M_c) > 0 \]

or locally, just near the equilibria:

\[ \mu - M_c < 0 \quad \mu - M_c > 0 \]

For the subcritical pitchfork, you should practice drawing an analogous picture on your own, starting with the "normal form" picture shown at the bottom of p. 20-5. See also Ex. 8.1.3 on p. 247 of Strogatz.

\[ (2) \text{ Bifurcations where the number of equilibria changes in conservative systems.} \]

**Observation:** In conservative systems, the only equilibria possible are saddles and centers (see Section 2 of Lecture 18). Therefore, saddle-node and pitchfork bifurcations are not possible in conservative systems.
Instead of a pair saddle + node, in conservative systems a pair saddle + center is either born or destroyed at a bifurcation.

**Ex. 2** Bifurcation in the Duffing equation

\[ \ddot{x} + \mu x + x^3 = 0 \]

(see Ex. 1 in lecture 18).

**IMPORTANT NOTE:** The closed orbits here are **not** limit cycles. Limit cycles are isolated closed orbits that either attract or repel all nearby trajectories. Limit cycles exist only in systems with energy dissipation and gain. In contrast, in conservative systems, there is a continuum of closed orbits around a center equilibrium, and their amplitudes are determined only by the initial condition.
Ex. 3. Bifurcation in the Kapitza pendulum

\[ \ddot{\theta} = -\sin \theta \cdot (1 + \mu \cos \theta), \quad \mu > 0 \]

(see Problem 8 in HW 13).

When \( \mu < 1 \), the equilibria are at \( \theta = \pi n \), as in the simple pendulum. However, when \( \mu > 1 \), two other equilibria are born:

\[ \theta_0 = \arccos \left( -\frac{1}{\mu} \right) \]

(see the figure of the pendulum on the left). The bifurcation of the phase portrait is the following:

\[ \mu < 1 \]

\[ \mu > 1 \]

(3) Bifurcations where the type of the equilibrium changes. Hopf bifurcation.

\[ \square \] Uninteresting case: 2D transcritical bifurcation

Normal form:

\[ \dot{x} = \mu x - x^2 \]
\[ \dot{y} = -y \]

(5)
The situation here is very similar to that for 2D saddle-node and pitchfork bifurcations.

The interesting case: Hopf bifurcation.

In Section 5 of Lecture 17 we showed that if we slightly change coefficients in system (1), then the only type of equilibrium that may change is the center. In the complex plane of the eigenvalues, the situation is:

\[ \text{Im}\lambda \quad \text{Re}\lambda \quad \text{or} \quad \text{Im}\lambda \quad \text{Re}\lambda \]

center becomes unstable spiral  center becomes stable spiral

More generally, as \( \mu \) changes through the critical value \( \mu_c \), a Hopf bifurcation can occur, which is manifested by two things:

- a stable spiral becomes an unstable spiral, and
- a stable limit cycle is born near the equilibrium.
The bifurcation diagram for the Hopf bifurcation is shown on the left. This diagram looks similar to the diagram of the supercritical pitchfork bifurcation, and indeed, the Hopf bifurcation is equivalent to the supercritical pitchfork bifurcation in polar coordinates. This is shown by the normal form of the Hopf bifurcation in polar coordinates:

\[
\begin{align*}
\dot{r} &= (\mu - \mu_c) r - r^3 \\
\dot{\theta} &= 1.
\end{align*}
\] (6)

(More generally, \(\dot{\theta}\) can be any function without zeros.) The phase portrait of the 1st equation of 6 is:

\[\begin{align*}
M < \mu_c & : & r^* = (\mu - \mu_c) r - r^3 \\
\text{stable spiral} & & \text{unstable spiral}
\end{align*}\]

\[\begin{align*}
M > \mu_c & : & r^* = (\mu - \mu_c) r - r^3 \\
\text{stable limit cycle} & & \text{unstable limit cycle}
\end{align*}\]
In particular, the radius of the limit cycle emerging at the Hopf bifurcation near the critical value of $\mu$ is:

$$r_0 = \sqrt{\mu - \mu_c}.$$  \hspace{1cm} (7)

An example of a system undergoing Hopf bifurcation is the van der Pol oscillator with small $\mu$:

$$\ddot{x} + \mu \dot{x} (x^2 - 1) + x = 0.$$  \hspace{1cm} (8)

The limit cycle of (8) was found in Ex. 3 of Lecture 19. See, however, the analysis of a seeming paradox related to condition (7) applied to that limit cycle: Ex. 8.4.1 on p. 264 of Strogatz.

Ex. 4  Not a Hopf bifurcation

The type of the equilibrium can also change from a stable spiral to an unstable spiral without creating a stable limit cycle. Then such a change is not a Hopf bifurcation. An example is the harmonic oscillator with damping or amplification:

$$\ddot{x} + \mu \dot{x} + x = 0.$$  \hspace{1cm} (9)

$\mu > 0$ The equilibrium $(0,0)$ is a stable spiral (Problem 6 in HW 13).
\( \mu < 0 \) the equilibrium is an unstable spiral. However, there is no limit cycle in Eq. (9) because it has only linear damping or amplification. In other words, the percentage amount of energy lost or gained does not depend on the amplitude of the oscillations. Clearly then, there is no reason why oscillations with a particular amplitude must be selected as a limit cycle.

So, to have a limit cycle, the system must have nonlinear damping and amplification, whereby the percentage amount of energy lost or gain should be sensitive to the amplitude of the oscillations, as it is for the van der Pol oscillator (8).

The Hopf bifurcation can also be subcritical. A subcritical Hopf bifurcation can lead to the solution "jumping" from the zero equilibrium to a stable limit cycle of finite radius. This should be contrasted to the supercritical Hopf bifurcation, where the limit cycle near the bifurcation has a small amplitude, see (7).

**Supercritical Hopf:**

\[
\frac{r = \mu r - r^3}{\text{Radius of cycle}}
\]

- Stable
- Unstable

**Subcritical Hopf:**

\[
\frac{r = \mu r + r^3 - r^5}{\text{Radius of cycle}}
\]

- Stable
- Unstable
Review the discussion of the subcritical pitchfork bifurcation in Lecture 16 and Problem 5 in HW 14, as well as pp. 251-252 in Strogatz about the subcritical Hopf bifurcation.

To conclude the discussion about bifurcations, we may ask: What happens if in Eq. (6), $\dot{\theta} \neq 1$ but

$\dot{\theta} = g(\theta, r)$ s.t. $g(\theta, r) = 0$ for some $\theta, r$?

For the answer, see the figure on p. 262 of Strogatz.