Lecture 2.
Review of undergraduate Linear Algebra

1. Matrix multiplication.

Let \( A = (p \times q) \), \( B = (q \times s) \), \( A_{ik} \) = element of \( A \) in the \( i \)th row \& \( k \)th column.

Then

\[
(AB)_{ij} = \sum_{k=1}^{q} A_{ik} B_{kj}.
\]

If we write:

\[
A = \begin{bmatrix}
\vec{a}_1 \\
\vec{a}_2 \\
\vdots \\
\vec{a}_p
\end{bmatrix}, \quad B = \begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_s
\end{bmatrix}_f
\]

then

\[
AB = \text{row-by-column} \quad \text{row}
\]

\[
= \begin{bmatrix}
\vec{a}_1 B_1 & \vec{a}_1 B_2 & \ldots & \vec{a}_1 B_s \\
\vec{a}_2 B_1 & \vec{a}_2 B_2 & \ldots & \vec{a}_2 B_s \\
\vdots & \vdots & \ddots & \vdots \\
\vec{a}_p B_1 & \vec{a}_p B_2 & \ldots & \vec{a}_p B_s
\end{bmatrix}
\]

each entry is a scalar

\[
= \begin{bmatrix}
AB_1 \\
\underline{AB_2} \\
\underline{\vdots} \\
\underline{AB_s}
\end{bmatrix}
\]

\( i \)th column of \( AB \).

Properties:

- associativity: \( (AB)C = A(BC) \).
- in general, matrices don't commute: \( AB \neq BA \) in general.
- but any matrix commutes with a scalar: \( \alpha A = A \alpha \).
\( AB = AC \quad \Rightarrow \quad B = C \)

E.g.:
\[
\begin{pmatrix}
1 & 3 \\
2 & 6
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 3 \\
2 & 6
\end{pmatrix}
\begin{pmatrix}
4 & 3 \\
-1 & 5
\end{pmatrix}
\]

A \quad B = \begin{pmatrix}
1 & 3 \\
2 & 6
\end{pmatrix} \quad \begin{pmatrix}
4 & 3 \\
-1 & 5
\end{pmatrix}
\]

As a corollary of the above,
\( AB = 0 \quad \Rightarrow \quad A \) or \( B = 0 \). (zero matrix)

2) Partitioned matrices (Sec. 1.5).

Matrices can be partitioned into blocks,

E.g.:
\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12
\end{pmatrix}
= \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
= \begin{pmatrix}
(2 \times 3) & (2 \times 1) \\
(1 \times 3) & (1 \times 1)
\end{pmatrix}
\]

If
\[
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}, \quad B = \begin{pmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23}
\end{pmatrix},
\]

\[
AB = \begin{pmatrix}
A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} & A_{11}B_{13} + A_{12}B_{23} \\
A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & A_{21}B_{13} + A_{22}B_{23}
\end{pmatrix}
\]

provided that all matrix products \( A_{ik}B_{kj} \) are defined.

A generalization when \( A \) consists of \( m \times n \) blocks is obvious.

E.g., take
\[
B = \begin{pmatrix}
0 & 1 & 2 & 3 \\
4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 \\
12 & 13 & 14 & 15
\end{pmatrix}
\]
3) Another view of matrix multiplication.

Consider first

\[
A \cdot x = \begin{bmatrix} A_1, A_2, \ldots, A_q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix} = \begin{bmatrix} (x_1) \\ (x_2) \\ \vdots \\ (x_q) \end{bmatrix}
\]

by block multiplication

\[
A_1 x_1 + A_2 x_2 + \ldots + A_q x_q
\]

This formula is extremely important and will be used heavily throughout the course.

In words: \( A \cdot x \) is a linear combination of the columns of \( A \).

\[
\text{E.g.: } (\frac{1}{2}, \frac{3}{4})(x_1) = \frac{1}{2}x_1 + \frac{3}{4}x_2 = \frac{5}{4}x_2.
\]

Now let \( A = \begin{bmatrix} A_1, \ldots, A_q \end{bmatrix}, B = \begin{bmatrix} B_1 \\ \vdots \\ B_q \end{bmatrix} \) \( \text{rows} \)

Then

\[
AB = A_1 B_1 + \ldots + A_q B_q
\]  

(\ref{AxB})

4) Solving a system of linear eqs. by Gauss-Jordan elimination.

\( \text{To solve } A \cdot x = b, \text{ we perform row reduction on } [A | b]. \)
Example 1

\[
\begin{bmatrix}
3 & -3 & 0 & 12 & 18 \\
1 & 0 & 2 & 7 & 6 \\
1 & -1 & 0 & 3 & 1 \\
-2 & 3 & 2 & -3 & -2 \\
\end{bmatrix}
\begin{bmatrix}
-9 \\
-2 \\
-5 \\
11 \\
\end{bmatrix}
\]

A \quad b

Operations

1) row1 \rightarrow row1/3
row2 \rightarrow row2 - \frac{row1}{3}
row3 \rightarrow row3 - \frac{row1}{3}
row4 \rightarrow row4 + \frac{2}{3} row1

Result:

\[
E_1 A = \begin{bmatrix}
1 & -1 & 0 & 4 & 6 & -3 \\
0 & 1 & 2 & 3 & 0 & 1 \\
0 & 0 & 0 & 1 & 5 & 2 \\
0 & 1 & 2 & 5 & 10 & 5 \\
\end{bmatrix}
\]

In matrix form (HW2):

\[
\begin{bmatrix}
\frac{1}{3} & 0 & 0 & 0 \\
-\frac{1}{3} & 1 & 0 & 0 \\
-\frac{1}{3} & 0 & 1 & 0 \\
\frac{2}{3} & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
A \\
E_1 \\
\end{bmatrix}
\]

2) row4 \rightarrow row4 - row2

Result:

\[
E_2 \cdot (E_1 A) = \begin{bmatrix}
1 & -1 & 0 & 4 & 6 & -3 \\
0 & 1 & 2 & 3 & 0 & 1 \\
0 & 0 & 0 & 1 & 5 & 2 \\
0 & 0 & 0 & 2 & 10 & 4 \\
\end{bmatrix}
\]
3) \( \text{row 4} \rightarrow \text{row 4} - 2 \cdot \text{row 3} \)

\[
E_3 \cdot (E_2E_1A) = \begin{pmatrix} 1 & -1 & 0 & 4 & 6 & -3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

Result:

\[
E_3 \cdot (E_2E_1A) = \begin{pmatrix} 1 & -1 & 0 & 4 & 6 & -3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

4) \( \text{column 3} \leftrightarrow \text{column 4} \)

(meaning: rename \( x_3 \leftrightarrow x_4 \))

\[
(E_3E_2E_1A)E_4 = \begin{pmatrix} 1 & -1 & 4 & 0 & 6 & -3 \\ 0 & 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
\text{Intermediate result: (key thm. B.53)}.
\]

\[
\begin{pmatrix} (E_3E_2E_1 \cdot A) \\ (E_3E_2E_1 \cdot A) \cdot E_4 \end{pmatrix} = \begin{pmatrix} \ast \ast \ast \ast \ast \end{pmatrix} \quad \text{(upper-\( \Delta \) matrix \( U \))}
\]

\( E_1, E_2, E_3 \) are lower-\( \Delta \).

The product of lower-\( \Delta \) matrices is lower-\( \Delta \) (\#1 of HW2). Thus \( E_3E_2E_1 = L^\sim \) (some lower-\( \Delta \) matrix).

Thus \( L^\sim A = U \) for "any" \( A \).
Note: If in our example we had the 3rd row of A equal the 2nd row, then at step 2) we would have obtained

\[ E_2 E_1 A = \begin{pmatrix} 1 & -1 & 0 & 4 & 6 & -3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 4 \end{pmatrix}. \]

Then to get the form \((\text{REF})\), we would have needed to interchange the 3rd and 4th row.

Therefore, above, the "any" means "any, up to a row permutation".

Return to our example

5) We now back-substitute (start with the last nonzero row).

\[
\begin{align*}
\text{row 2} & \rightarrow \text{row 2} - 3 \times \text{row 3}, \\
\text{row 1} & \rightarrow \text{row 1} - 4 \times \text{row 3}
\end{align*}
\]

\[
\begin{pmatrix} 1 & -1 & 0 & 0 & 14 & 11 \\ 0 & 1 & 0 & 0 & -15 & -5 \\ 0 & 0 & 1 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

6) \text{row 1} \rightarrow \text{row 1} + \text{row 2}

\[
\begin{pmatrix} 1 & 0 & 0 & 0 & -29 & -16 \\ 0 & 1 & 0 & 0 & -15 & -5 \\ 0 & 0 & 1 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\{ \text{REF or Gauss-reduced form} \}

\[
\begin{align*}
x_1 & = -16 + 29x_5 \\
x_2 & = -5 + 15x_5 \\
x_3 & = 2 - 5x_5 \\
x_4, x_5 & = \text{free variables} \\
x_1, x_2, x_3 & = \text{leading variables}
\end{align*}
\]

Switch \(x_3 \leftrightarrow x_4\) to get to the original
5) **Matrix inverses**

Let $A$ be a square matrix. Then under certain conditions (to be specified later) there is a matrix $A^{-1}$, called the inverse of $A$, s.t.

$$A^{-1}A = I = AA^{-1},$$

where $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ is the $p \times p$ identity matrix.

**Properties of inverses:**

- $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$.

- If both $A$ and $B$ are $p \times p$ and have inverses, then so does $AB$ and $(AB)^{-1} = B^{-1}A^{-1}$. **Note the order!**

**Proof:** It suffices to verify that $B^{-1}A^{-1} (AB) = I$.

Indeed:

$$\begin{align*}
(B^{-1}A^{-1})AB &= B^{-1}(A^{-1}A)B \\
&= B^{-1}IB = B^{-1}B = I
\end{align*}$$

**Note:** If $A^{-1}$ exists, then $Ax = b$ can be solved as $x = A^{-1}b$.

However, this is mostly a theoretical but not practical tool, since to actually compute $x$, one uses the Gauss-Jordan elimination rather than compute $A^{-1}$. 
Note 2: If \( A = (p \times q) \) is not square, then one can define either (but not both!) its left or right inverse, i.e.,
either \( LA = I_{q \times q} \) or \( AR = I_{p \times p} \).
However, we will use alternative tools to describe the solution of \( Ax = b \)
when \( A = p \times q \).

\( \textcircled{6} \) Linear independence of vectors.

Def. A set of vectors \( v_1, \ldots, v_q \) is called linearly independent (l.i.) if the only solution \( (c_1, \ldots, c_q) \) of the vector equation
\[
    c_1 v_1 + c_2 v_2 + \ldots + c_q v_q = 0 \tag{**A**}
\]
is
\[
    c_1 = c_2 = \ldots = c_q = 0.
\]
Otherwise, the set is called linearly dependent (l.d.). Then, (**A**) holds when some of the \( c \)'s are \( \neq 0 \).

Meaning of linear dependence
Suppose one of the \( c \)'s \( \neq 0 \), e.g., \( c_1 \neq 0 \). Then divide by \( c_1 \):
\[
    v_1 = \left( \frac{c_2}{c_1} \right) v_2 + \left( \frac{-c_3}{c_1} \right) v_3 + \ldots + \left( \frac{-c_q}{c_1} \right) v_q.
\]
Thus, all vectors are l.d. iff one of them is a linear combination of the others.
In \( \mathbb{R}^2 \), two vectors are \textit{l. i.} if they are not \( \perp \):

\[
\begin{align*}
\vec{v}_1 & \quad \text{(l. i.)} \\
\vec{v}_2 & \\
\end{align*}
\]

Any three vectors in \( \mathbb{R}^2 \) are \textit{l. d.}:

\[
\begin{align*}
\vec{v}_1 & \\
\vec{v}_2 & \\
\vec{v}_3 & = c_1 \vec{v}_1 + c_2 \vec{v}_2.
\end{align*}
\]

In \( \mathbb{R}^3 \), three vectors are \textit{l. i.} if they do \textbf{not} lie in the same plane, i.e., if

\[
\vec{v}_3 \neq c_1 \vec{v}_1 + c_2 \vec{v}_2.
\]

\textbf{Question:} How can we determine if a set is \textit{l. i.}?

\[
\begin{align*}
\vec{v}_1 c_1 + \cdots + \vec{v}_9 c_9 &= 0 \\
\begin{bmatrix}
\vec{v}_1, \cdots, \vec{v}_9
\end{bmatrix}
\begin{pmatrix}
c_1 \\
\vdots \\
c_9
\end{pmatrix} &= \begin{pmatrix} 0 \\
\vdots \\
0
\end{pmatrix}
\end{align*}
\]

(see Ex. 1)

\[
\begin{align*}
\text{Transform } \begin{bmatrix}
\vec{v}_1, \cdots, \vec{v}_9
\end{bmatrix}
\text{ to REF}. & \quad \text{If there are free variables, then the set is l.d. If there are no free variables, then the set is l. i.}
\end{align*}
\]

And hence \( \text{c} \neq \text{c0} \).
Note: A zero vector is l.d. on any set of vectors. Indeed, let \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_q \} \) be some set of vectors. Obviously,

\[ 0 = \mathbf{v}_1 + \cdots + 0 \cdot \mathbf{v}_q + c \cdot 0 = \mathbf{0}, \]

\( \{ \mathbf{v}_1, \ldots, \mathbf{v}_q, \mathbf{0} \} \) is a l.d. set. ✓
Example 2. Show formally that \( \{(\frac{1}{2}), (\frac{3}{4}), (\frac{5}{6})\} \) is l.d.

Solution: Solve \( (2) c_1 + (3) c_2 + (6) c_3 = (0) \) \( \Rightarrow \)

\[
\begin{bmatrix}
1 & 3 & 5 & | & 10 \\
2 & 4 & 6 & | & 0
\end{bmatrix}
\text{ REF } \rightarrow 
\begin{bmatrix}
1 & 0 & 1 & | & 0 \\
0 & 1 & 0 & | & 0
\end{bmatrix}
\]

\[ c_1 - c_3 = 0 \Rightarrow c_1 = c_3 \]
\[ c_2 + 2c_3 = 0 \Rightarrow c_2 = -2c_3 \]  \( \Rightarrow \) \( c_3 = \text{free} \)

Then \( \frac{1}{2} c_3 + \frac{3}{4} (-2c_3) + \frac{5}{6} (c_3) = (0) \).

Since \( c_3 = \text{free} \), it can be chosen to be 1.

Then: \( \frac{1}{2} = (\frac{3}{4}) \cdot 2 + (\frac{5}{6}) \cdot (-1), \)

We have shown that the first vector is a lin. combination of the other two, \( \Rightarrow \) the set is l.d.

Note: On the contrary, the set \( \{(\frac{1}{2}), (\frac{3}{4})\} \) is l.d. i.e., Indeed,

\[
\begin{bmatrix}
1 & 3 & 5 & | & 10 \\
2 & 4 & 6 & | & 0
\end{bmatrix}
\text{ REF } \rightarrow 
\begin{bmatrix}
1 & 0 & 1 & | & 0 \\
0 & 1 & 0 & | & 0
\end{bmatrix}
\]

is the only solution of \( (2) c_1 + (3) c_2 = (8) \).

Corollary: If \( \{v_1, \ldots, v_q\} \) is a set of \( p \)-dimensional vectors, and if \( q > p \), then this set is l.d.
7 Singular and nonsingular matrices

Def: A square $p \times p$ matrix $A$ is **nonsingular** if the **only solution** of $Ax = 0$ is the **trivial solution** $x = 0$.

Equivalently, $A$ is **singular** if there is a $x \neq 0$ s.t. $Ax = 0$.

Note: If $x$ solves $Ax = 0$, so does $ax$ for any scalar $a$. Indeed:

$A(ax) = a(Ax) = a \cdot 0 = 0$.

Thm. 1: If $A$ is $p \times p$, the following statements are equivalent:

1. $A$ is nonsingular.
2. All columns of $A$ are l. i.
3. A unique inverse, $A^{-1}$, exists.
4. $Ax = b$ has a unique solution for any $b$.

This theorem answers Q1 of Lecture 1.

Question: What are the possibilities regarding the number of solutions of $Ax = b$?

Example 7: Let $v_1, v_2, b$ be vectors in $IR^2$. Consider the system
\[ \begin{bmatrix} v_1, v_2 \end{bmatrix} (c_1, c_2) = b, \text{ or } v_1 c_1 + v_2 c_2 = b. \]

There are 3 possibilities:

1) \( v_1 \neq v_2 \), \( b \) arbitrary

A unique \((c_1, c_2)\) always exists.

This agrees with Thm. 1.

(See Sec. 6 of this lecture)

2) \( v_1 \parallel v_2 \), \( b \parallel v \)

No lin. combination of \( v_1, v_2 \) would yield \( b \).

No solutions

3) \( v_1 \parallel v_2 \parallel b \)

\[ \begin{align*}
3v_1 + 0 \cdot v_2 &= b \\
v_1 - v_2 &= b \\
0 \cdot v_1 - \frac{3}{2} \cdot v_2 &= b
\end{align*} \]

\[ b = 3v_1 \]

\[ v_2 = -2v_1 \]

\[ \text{infinitely many sol'n's} \]

Ex. 3 illustrated the general case: there are only 3 possibilities about the \# of solutions of \( Ax = b \): (1) unique sol'n, (2) no sol'n's, (3) \( \infty \) many solutions.
Thm. 2 (Key Thm. 4.16; addresses cases (1) and (3) above).

Let $x_0$ be some particular solution of $Ax = b$, i.e., $Ax_0 = b$. Let $h$ range over the set of all possible solutions of the homogeneous problem $Ah = 0$. Then the set of all possible solutions of $Ax = b$ is given by $x = x_0 + h$.

Proof: Given: $Ax_0 = b$. Let $x$ be another solution, $\Rightarrow Ax = b$.

Subtract 1st eqn. from 2nd:

$$Ax - Ax_0 = b - b$$
$$A(x - x_0) = 0 \Rightarrow x - x_0 = h$$
$$x = x_0 + h$$

q.e.d.

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Corollary (Key Thm. Corollary 4.19, or Fredholm Alternative).

Consider $Ax = b$ with $A = (p \times q)$ (this includes the case $(p \times p)$ if $p = q$.) Then either $Ax = b$ has a unique sol'n for any $b$ (i.e., $A = (p \times p)$ and nonsingular by Thm. 1) or $Ax = 0$ has $\infty$ many sol'n's, but not both.
Discussion: The 2nd possibility means that if $Ax = 0$ has nontrivial solutions $h \neq 0$, then $Ax = b$ may have either 0 or $\infty$ many solutions (see Ex. 3).

Note: Later we will establish an algebraic condition that will tell us w/o solving $Ax = b$ whether this eqn. has 0 or $\infty$ many solutions.

8. Rank of a matrix
The concept of linear dependence/independence of vectors is very important because it is closely related to how many solutions $Ax = b$ can have. Later we will see many other facets where this concept is important. Hence:

Def: The rank of a matrix is the number of its lin. independent columns.

Ask Q: What's the rank of $\begin{pmatrix} 1 & 3 & 8 \end{pmatrix}$?

Restatements of facts from Thms. 18.2 in terms of rank

(1) $A = (p \times p)$ is singular iff rank $(A) < p$.

(This is just Thm. 1).

(2) If $A = (p \times q)$ and rank $(A) < q$, then $Ax = b$ does not have a unique solution (i.e. can have 0 or $\infty$ many solutions).
Proof: \( \text{rank}(A) < q \Rightarrow \) columns of \( A \) are l.i.d.

\[ \exists x \neq 0 \text{ s.t.} x_1 A_1 + x_2 A_2 + \ldots x_q A_q = 0, \quad \Rightarrow \]

\[ A \cdot x = 0 \text{ for } x \neq 0. \]

Thus, the homogeneous eqn. \( A \cdot x = 0 \) has nontrivial solutions, \( \Rightarrow \) by the Fredholm Alternative (Corollary to Thm. 2) \( A \cdot x = b \) does not have a unique soln. \( \quad \text{q.e.d.} \)

**Thm. 3** In any \( p \times q \) matrix \( A \),

the \# of l.i. columns = \# of l.i. rows.

Proof (sketch):

1) Recall Ex. 1, where we showed that

\[ (E_3E_2E_1) \cdot A \cdot E_4 = \begin{pmatrix} \star & \star \\ \star & \star \\ \end{pmatrix} \]

\[ U \]

The elementary row operations \((E_1, E_2, E_3)\) and column interchanges \((E_4)\) do not alter lin. dependence or independence of rows and columns. (Here is the justification is missing).

Therefore, the \# of l.i. rows and columns of \( A \) are the same as those of \( U \).

2) By inspection, \( U \) has 3 l.i. rows (recall the Note at the end of Sec. 6 of this lecture, where we showed that a zero row is always l.i. on nonzero rows).
3) $U$ has 3 l.i. columns: 1st, 2nd, 3rd. Indeed, the 4th and 5th columns are l.i. on them:

$$\begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, for this $U$, # of l.i. columns = # of l.i. rows. This is guaranteed by the diagonal structure of $U$:

$$\begin{pmatrix} 1 & 0 & \ast \\ 0 & 1 & \ast \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix}.$$

The same will apply to any other matrix $A$.

Corollary 1 Let $A = (p \times q)$.

Then $\text{rank}(A) \leq \min(p, q)$.

Corollary 2 If $A = (p \times q)$ and $p < q$ (more unknowns than eqs.),

then $Ax = 0$ has many solutions.

Proof: $\text{rank}(A) \leq p < q$, then repeat the proof of Restatement 2 above.
Thm. 4 ( #5 of Sec. 4.4)

Let $A$ be $p \times p$ and nonsingular, and let $B = (p \times q)$. Prove: $\text{rank}(B) = \text{rank}(AB)$.

Proof: posted on the website.
Read on your own.

Corollary 1 If both $A$ and $B$ are $p \times p$ and nonsingular, then $AB$ is nonsingular.

Corollary 2 If $\text{rank}(B) = k$, $B = (p \times q)$, and $A = (q \times q)$ and nonsingular, then $\text{rank}(BA) = \text{rank}(B)$.

You will prove both corollaries in a HW problem.

Note that Thm. 4 + Corollary 2 can be combined:

A rank of a matrix is not changed if it is multiplied by a nonsingular matrix.

9) Range and null-space.

Let $A = [A_1, \ldots, A_q]$. Then:

Def. Range of $A = \mathcal{R}(A)$ is the set of all possible linear combinations of the columns of $A$. I.e., any

$Ax = x_1A_1 + \ldots + x_qA_q$ is in $\mathcal{R}(A)$. 
Note that if \( Ax = b \) has a solution, then \( b \in \mathbb{R}(A) \), since \( b = A_1 x_1 + \ldots + A_q x_q \).

**Def:** The set of all solutions of \( Ax = 0 \) forms the null space of \( A \), \( N(A) \).

**(10)** Transpose of a matrix

Let \( A = (p \times q) \). Then \( A^T = (q \times p) \) with:
- rows of \( A \) = columns of \( A^T \),
- columns of \( A \) = rows of \( A^T \).

**E.g.,** \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \), \( A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \).

**Properties of the transpose**

- \( (A + B)^T = A^T + B^T \)
- \( (AB)^T = B^T A^T \) (note the order!)
- \( (A^T)^T = A \)
- \( (A^{-1})^T = (A^T)^{-1} \)

**Restatment of Thm. 3:**

\[ \text{rank}(A) = \text{rank}(A^T) \]

**Note:** We will see that \( A^T \) plays an important role in the solution of \( Ax = b \) and \( \frac{dx}{dt} = Ax + u \).
When entries of $A$ are complex, then instead of $A^T$ one often uses
\[ A^H = (A^*)^T \]
\[ \text{c.c. & transpose} \]
$A^H$ is called the Hermitian conjugate of $A$ (or sometimes the adjoint of $A$).

Def: $A$ is symmetric if $A = A^T$.
E.g., \( \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \) is symmetric.

$A$ is Hermitian if $A = A^H$.
E.g., \( \begin{pmatrix} 1 & 2-3i \\ 2+3i & 4 \end{pmatrix} \) is Hermitian.

Of course, if $A$ is real, then $A^T = A^H$.

- Hermitian matrices have special properties, as we will see later.

Remark: In most parts of this course we will use real matrices, so will refer to $A^T$.
When a matrix is complex, simply substitute $A^H$ for $A^T$.

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11. Determinants (Secs. 4.5 § & 4.6).

Note: Determinants are useful for theoretical representation of the solutions, but not for their computation. (See Table 4.142 on p. 169 + a few problems.)
\[ \text{2x2 matrices} \]
\[
\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]

\[ \text{3x3 matrices} \]
\[
\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot A_{11} + a_{12}A_{12} + a_{13}A_{13}
\]
\[= a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \cdot (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \]

Cofactor \( A_{ij} \) is the determinant of a matrix obtained from \( A \) by crossing out the \( i \)-th row and \( j \)-th column, and multiplied by \((-1)^{i+j}\).

In fact, instead of the first row, one can use the cofactor expansion w.r.t. any row or column:
\[
\det A = \sum_{k=1}^{p} a_{ik} A_{ik} \quad \text{for any } i=1, \ldots, p
\]

(\( p \times p \))

Properties of determinants:

\[ \text{Thm. 5 (Corollary (4.31)(a))} \]
\[
\det A = \det A^T
\]
\[\text{but } \det A^H = \det A \]

Thm. 6. (Thm. (4.35)(a))

A is nonsingular iff \( \det(A) \neq 0 \).

Thm. 7 (Thm. (4.36))

\[ \det(AB) = \det(A) \cdot \det(B) \]

Proofs — see book. (You don't need to memorize the details, but it is instructive to see them)

You must also read and know the results expressed in Corollary (4.31)(a-d) & Thm. (4.32).

Useful property: Let \( T \) be either an upper-\( \Delta \) or a lower-\( \Delta \) matrix, with diagonal entries \( t_{11}, t_{22}, \ldots, t_{pp} \). Then

\[ \det(T) = t_{11} \cdot t_{22} \cdot \ldots \cdot t_{pp} \]

In particular, if any of \( t_{jj} = 0 \), then \( T \) is singular.

(Proof — a HW problem.)

Thm. 8 (Cramer's Rule); Thm. (4.49)

Let \( A \) be nonsingular. The entries \( x_i \) of the solution \( x \) of \( Ax = \vec{b} \) are given by

\[ x_i = \Delta_i / \Delta, \]

where \( \Delta = \det(A) \) and \( \Delta_i = \det \) of a matrix obtained from \( A \) by replacing its \( i \)-th column with \( \vec{b} \).

(Proof — bonus HW problem.)
Example 4

Solve \((\begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}\).

Solution:

\[
\begin{align*}
    x_1 &= \frac{\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = -1 \\
    x_2 &= \frac{\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = 2.
\end{align*}
\]

Determinants can also be used for the theoretical representation of the inverse (but not for its computation!).

Thm. 10 (Corollary (4.41)).

If \(A\) is nonsingular, then

\[
A^{-1} = \frac{\text{adj} A}{\det A},
\]

where \(\text{adj} A\), the adjugate of \(A\), is a matrix s.t.

\[(\text{adj} A)_{ij} = A_{ji}, \text{ the } (j,i)\text{-th cofactor of } A\].