Lecture 18  Sketching global phase portraits of nonlinear systems.

1. General rules of sketching a phase portrait
   \[ \begin{align*}
   \dot{x} &= f(x, y) \\
   \dot{y} &= g(x, y)
   \end{align*} \] (1)

   - Find all the equilibria. Use linearization around them to determine their types. Sketch the local phase portrait near each equilibrium.
   - It may be helpful to sketch the directions of the trajectories at the two nullclines, \( f(x, y) = 0 \) and \( g(x, y) = 0 \). Specifically:
     - on \( f(x, y) = 0 \), \( \dot{x} = 0 \), \( \Rightarrow \) the trajectory must be vertical (up or down);
     - on \( g(x, y) = 0 \), \( \dot{y} = 0 \), \( \Rightarrow \) the trajectory must be horizontal (left or right).
   - It may also be helpful to sketch the directions of the trajectories on the axes \( x = 0 \) and \( y = 0 \).
   - Trajectories cannot intersect other than at the equilibria.
   - There should be no empty regions in the phase plane: any point \((x_0, y_0)\) (in the domains of \( f \) and \( g \)) can be an initial condition for (1).
Before we practice sketching phase portraits for systems (1) with \( f \) and \( g \) of general form, we will learn about a special subclass of systems (1).

(2) **Conservative systems, Potential energy, and the total energy conservation.**

Many model problems come from Newtonian mechanics of systems with 1 degree of freedom (e.g., a particle whose motion is restricted to be along a line). When there is no energy dissipation in such a model, the equation often is:

\[
mx'' = -\frac{\partial V(x)}{\partial x},
\]

where \( V(x) \) is the potential energy of the system. For example, for the mass on a spring, \( V(x) = kx^2/2 \), and then (2) yields the familiar equation

\[
mx'' = -kx.
\]

Systems of the form (2) are called conservative because they conserve the total energy:

\[
E_{\text{total}} = E_{\text{kinetic}} + E_{\text{potential}} = \frac{mx^2}{2} + V(x) = \text{const.}
\]

This can be shown using Eq. (2) as follows.
\[ \ddot{x} \cdot \left( m \ddot{x} = -\frac{\partial V}{\partial x} \right) \Rightarrow m \frac{d^2x}{dt^2} \cdot \frac{dx}{dt} = -\frac{\partial V(x)}{\partial x} \cdot \frac{dx}{dt} \tag{4} \]

\[ \Rightarrow \frac{m}{2} \frac{d}{dt} \left( (dx/dt)^2 \right) = -\frac{d}{dt} V(x) \]

\[ \Rightarrow \frac{d}{dt} \left( \frac{m \dot{x}^2}{2} + V(x) \right) = 0 \Rightarrow \frac{m \dot{x}^2}{2} + V(x) = \text{const}, \]

which is (3).

Note that the conservation of the total energy is intuitively expected because, as we said earlier, in systems described by (2), all dissipative effects are neglected.

Let us now show that:

**Thm. 1** In a conservative system, an isolated equilibrium with nonzero eigenvalues can only be either a saddle or a center. In particular, attracting equilibria - the stable node and the stable spiral - cannot exist in conservative systems.

**Proof:** Let \( x_* \) be an equilibrium of (2). That equation can be equivalently rewritten as

\[ \begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\frac{\partial V}{\partial x} = g(x) & \text{.} \tag{5}
\end{align*} \]

The linearization of (5) near \( x_* \) is (see Eq. (7) in Lecture 17):

\[ \begin{align*}
\tilde{x} &= x - x_*, & \tilde{y} &= y - 0 = y, & \text{and}
\end{align*} \]
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-g_x(*) & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}.
\] (6)

The eigenvalues of the Jacobian matrix in (6) are:
\[
\lambda = \pm \sqrt{g_x(*)} = \pm \sqrt{-\frac{\partial^2 V(x)}{\partial x^2}}.
\] (7)

Since \( g(x) \) is real, the eigenvalues can be either both imaginary (\( V_{xx}(*) > 0 \), the equilibrium is a center) or both real and of opposite sign (\( V_{xx}(*) < 0 \), the equilibrium is a saddle).

3) Examples of phase portraits of conservative systems, separatrices.

Ex. 1 (Duffing equation).
Sketch the phase portrait of
\[
\ddot{x} = x - x^3.
\] (8)

1) Find the equilibria and sketch the local phase portraits near them.

First, we rewrite (8) as
\[
\begin{align*}
x &= \dot{y} \\
y &= x - x^3 = g(x).
\end{align*}
\] (8')

a) The equilibria are found from:
\[ \begin{cases} y = 0 \\ x - x^3 = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x(1-x^2) = 0 \end{cases} \Rightarrow \]

there are 3 equilibria: \((-1,0), (0,0), (1,0)\).

b) Linearization. Using \(g_x(x) = 1 - 3x^2\), we have the following Jacobians:

@ \((-1,0)\)
\[
A = \begin{pmatrix} 0 & 1 \\ g_x(-1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \Rightarrow
\]

\(\lambda = \pm \sqrt{-2} = \pm i\sqrt{2}\), \(\Rightarrow\) the equilibrium is a \underline{center}.

@ \((0,0)\)
\[
A = \begin{pmatrix} 0 & 1 \\ g_x(0) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow
\]

\(\lambda_1 = +1, \ \nu_1 = (1); \ \lambda_2 = -1, \ \nu_2 = (-1)\). \underline{Saddle}

@ \((1,0)\)
\[
A = \begin{pmatrix} 0 & 1 \\ g_x(1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \Rightarrow
\]

\(\lambda = \pm i\sqrt{2}\), \underline{center}.

c) Local phase portraits.

\[ \begin{align*}
\dot{y} &= -2(x+1) \\
\dot{x} &= y > 0 \\
\Rightarrow \quad & y \downarrow.
\end{align*} \]

\[ \begin{align*}
\dot{y} &= -2x = -2(x-1) < 0 \\
\Rightarrow \quad & y \uparrow.
\end{align*} \]
2) Find the separatrix.

The separatrix is the trajectory that separates the phase portrait into different regions such that trajectories from one region never enter other regions. In section 3 of Lecture 17 we saw that the eigenvectors \( \mathbf{v}_1, \mathbf{v}_2 \) of the saddle are its separatrices. Here we'll show that they are actually parts of the same closed separatrix.

Let us show that for conservative systems, the equation of the separatrix can be found. Rewrite (8) in the form (2):

\[
\dot{x} = -\frac{3}{2x} \left(-\frac{x^2}{2} + \frac{x^4}{4}\right).
\]

Then, according to (4) with \( m=1 \), we have:

\[
\frac{\dot{x}^2}{2} \left(-\frac{x^2}{2} + \frac{x^4}{4}\right) = 0 \Rightarrow \\
\frac{x^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} = C = \text{const}
\]

Exin. Epotential. Etotal

The last equation defines a curve in the phase plane. Every such a curve is actually a trajectory, because the energy it is conserved. The separatrix in a conservative system is always the trajectory that passes through the saddle equilibrium. So, in our case its equation is:
\[
\frac{x^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} = \frac{0^2}{2} - \frac{0^2}{2} + \frac{0^4}{4} = 0,
\]

coordinates of the saddle (0,0)

or:

\[
\dot{x} = \pm \sqrt{x^2 - \frac{x^4}{2}}, \quad (9)
\]

This separatrix can be plotted either using Mathematica or Matlab, or using qualitative arguments. Namely, for \(x \approx 0\), \(x^4 \ll x^2\), and so \(\dot{x} \approx \pm \sqrt{x^2} \approx \pm x\), in agreement with the local phase portrait near the saddle (see p. 18-5).

On the other hand, for large \(|x|\), the separatrix does not exist, because the expression under the \(\sqrt{\cdot}\) in (9) becomes negative.

In fact, the separatrix exists up to \(|x|\) such that \(x^2 - x^{4/2} = 0 \Rightarrow 1 - x^{2/2} = 0\), \(\Rightarrow x = \pm \sqrt{2}\); for this \(x_{\text{max}}\), \(\dot{x} = \pm \sqrt{x^2 - x^{4/2}} = 0\).

Thus, the separatrix must look like this:

A separatrix that connects an equilibrium with itself is called a homoclinic orbit.
Combining the information from steps 1) and 2), we obtain the following phase portrait so far:

3) Finally, let us draw the direction of trajectories on the x- and y-axes. (In this case, drawing the directions on the nullclines $x - x^3 = 0$ i.e. on the vertical lines $x = 0$, $x = \pm 1$, yields no new important information.)

This is the global phase portrait of the Duffing equation (8).

Interpretation Equation (8) can be written as

$$\ddot{x} = -\frac{2}{\alpha} V(x) \quad \text{with} \quad V(x) = -\frac{x^2}{2} + \frac{x^4}{4}.$$
As we noted in lecture 13, such an equation describes — qualitatively, but not quantitatively — the motion of a particle of unit mass in a potential well of the shape shown below. In particular, the trajectories labeled A, B, C in the global phase portrait, correspond to the following motions:

Ex. 2 (Simple pendulum)
Sketch the global phase portrait of
\[ \ddot{x} = -\sin x, \quad \text{or} \]
\[ \begin{align*}
\dot{x} &= y \\
y &= -\sin x
\end{align*} \quad (10) \]

1) a) Find the equilibria:
In Ex. 3 of lecture 17 we found that they are \((\pi n, 0)\), where \(n\) is an integer. E.g., they are: \((-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0)\), etc.
b) Linearization.

In Problem 7 of HW 13 you show that \((-\pi, 0), (\pi, 0), (3\pi, 0), \ldots\) etc. are **saddles** with \(\lambda_1 = 1, \quad \nu_1 = (\ddagger), \quad \lambda_2 = -1, \quad \nu_2 = (\ddagger), \)
and \((0, 0), (2\pi, 0)\) etc. are centers with \(\lambda = \pm i\).

c) Local phase portraits:

You will justify the directions of the trajectories in the aforementioned Problem 7.

2) The **separatrix**.

The equation of the conservation of the total energy of the pendulum is:

\[
\dot{x} = \frac{2}{\sin(x)} (-\cos x) \quad \Rightarrow \quad \frac{\dot{x}^2}{\sin^2 x} = \text{const}. \tag{11}
\]

By definition, the **separatrix** passes through the saddle (e.g., \((\pi, 0)\)), whence

\[
\frac{\dot{x}^2}{2} - \cos x = \frac{o^2}{2} - \cos \pi = 1.
\]

Thus, the equation of the **separatrix** is

\[
\dot{x}^2 = 2(1 + \cos x). \tag{12}
\]
It can be either plotted with Matlab or Mathematica, or recognized as:

\[ x^2 = 2(1+\cos x) = 2 \cdot 2\cos^2 \frac{x}{2} = 4\cos^2 \frac{x}{2}, \]

so (12) is equivalent to:

\[ x = \pm 2\cos \frac{x}{2}. \]

Thus, the global phase portrait so far looks like this:

The separatrix that connects two different saddles is called a heteroclinic orbit (compare this with the definition of a homoclinic orbit on p. 18-7).

3) Finally, we will sketch the directions of the trajectories on the nullclines \( y = 0 \) and \( \sin x = 0 \), i.e., \( x = \pi n \):
This is the global phase portrait of the simple pendulum equation (10). The interpretation of the trajectories labeled A, B, C in terms of the motion of a particle in a potential well and also in terms of the actual pendulum, is below.

makes a full turn makes exactly oscillates in finite time and one turn in an about the keeps on rotating infinite time bottom equilibrium

4) Examples of phase portraits of nonconservative systems

Ex. 3 Sketch the phase portrait of a (slightly) damped pendulum:

\[ x'' + 2\gamma x' + \sin x = 0, \quad (13a) \]
or
\[ \begin{align*} x &= y \\ \dot{y} &= -\sin x - 2\gamma y \end{align*} \] \hfill (13b)

1) a) Find the equilibria:
\[ \begin{cases} y = 0 \\ -\sin x - 2\gamma y = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ \sin x = 0 \end{cases} \Rightarrow (\pi n, 0), \]
i.e. the same as for the pendulum without dissipation.

b) Linearization.

We can, of course, find the eigenvalues of the Jacobian matrix
\[
\begin{pmatrix} 0 & 1 \\ -\cos x & -2\gamma \end{pmatrix}
\]
at the equilibria, but we will use an easier method.

Recall our analysis in Section 5 of Lecture 17 of structural robustness of various types of equilibria. There, we found that all types of equilibria except for the center, are structurally robust. The center can become a stable spiral or an unstable spiral. In our case, we should expect the stable spiral because the dissipation (the term $2\gamma x$ in (13a)) will cause small oscillations near $x = 0$, $2\pi$, etc. to decay with time.
c) Thus, the local phase portraits of (13) should look like this:

\[ \begin{array}{c}
\text{\vector{\downarrow \quad \text{\nabla}}}
\end{array} \]

A saddle remains a saddle for a sufficiently small $\gamma$.

A center becomes a stable spiral, but the directions of trajectories remain the same for sufficiently small $\gamma$.

2) Separatrices for nonconservative systems cannot be found from the conservation of energy (because the energy is not conserved). However, there still is the question of how to connect the local separatrices of saddles with other parts of the phase portrait.

When we know that the nonconservative system was obtained from a conservative one by adding some sufficiently small dissipation, this connection should be done "by continuity" with the conservative case, as shown below.

\[ \begin{array}{c}
\text{\vector{\downarrow \quad \text{\nabla}}} \\
\end{array} \]
Namely, the trajectory coming out of the saddle \((-\pi, 0)\) to the right, must become one of the spirals winding into \((0,0)\). Similarly for the trajectory coming out of \((\pi, 0)\) to the left. Thus, the phase portrait so far is:

Next, the trajectories coming into the saddles are sketched following two rules stated at the beginning of this lecture:
- the trajectories do not intersect, and
- there should be no empty areas in the phase plane (assuming that the initial condition can be anywhere).

The result can be found by inspection and by some trial and error:
This is the global phase portrait of the damped pendulum (13). The step with drawing the directions at the nullclines \( y = 0 \) and \( -\sin x - 2xy = 0 \) or at the axes is not needed since we obtained similar results for the pendulum without dissipation in Example 2.

**Interpretation.** Although we cannot speak about undamped motion in the potential well, we can still draw analogies with the damped motion. So, the trajectories labeled \( A, B, C \) in the global phase portrait correspond to the following motions:

<table>
<thead>
<tr>
<th>( V(x) )</th>
<th>( A )</th>
<th>( V(x) )</th>
<th>( B )</th>
<th>( V(x) )</th>
<th>( C )</th>
</tr>
</thead>
</table>

- Pendulum has some initial velocity at \( x = -\pi \), so that it barely reaches \( x = \pi \) and stays there forever.
- Pendulum has some initial velocity at \( x = -\pi \), but not high enough to make it to \( x = \pi \); it oscillates near the bottom and eventually (at \( t \to +\infty \)) comes to rest there.
- Pendulum is barely pushed from \( x = -\pi \), oscillates near \( x = 0 \) and eventually comes to rest there.
Ex. 4 Sketch the phase portrait of the classic model of competing species (the Lotka-Volterra model).

\[ x = 3x - x^2 - 2xy \]

- growth of the isolated species
- intra-species competition (e.g., for food)
- inter-species competition

\[ y = 2y - y^2 - xy \]

See pp. 155-156 in Strogatz for a more thorough discussion.

The above equations are:

\[ x = x(3-x-2y) \]

\[ y = y(2-y-x) \]

(14)

1) a) Find the equilibria.

\[
\begin{cases}
  x(3-x-2y)=0 \\
y(2-y-x)=0
\end{cases} \Rightarrow \begin{cases} x=0 \text{ or } 3-x-2y=0 \\
y=0 \text{ or } 2-y-x=0
\end{cases}
\]

Case 1: \( x=0, y=0 \Rightarrow (x_*, y_*) = (0, 0) \).

Case 2: \( x=0, 2-y-x=0 \Rightarrow (x_*, y_*) = (0, 2) \).

Case 3: \( 3-x-2y=0, y=0 \Rightarrow (x_*, y_*) = (3, 0) \).

Case 4: \( 3-x-2y=0, 2-y-x=0 \Rightarrow (x_*, y_*) = (1, 1) \).

(solving this as a linear system for x, y)

b) Linearization.

The Jacobian is:
\[
\begin{pmatrix}
\frac{\partial^2}{\partial x^2} (3x-x^2-2xy) & \frac{\partial^2}{\partial y^2} (3x-x^2-2xy) \\
\frac{\partial^2}{\partial x \partial y} (2y-y^2-xy) & \frac{\partial^2}{\partial y^2} (2y-y^2-xy)
\end{pmatrix} = 
\begin{pmatrix}
3-2x-2y & -2x \\
-2y & 2-2y-x
\end{pmatrix}
\]

Then:

@ \( (0,0) \):
\[ A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \]
\[ \lambda_1 = 3, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = 2, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \Rightarrow \text{unstable node} \]

@ \( (0,2) \):
\[ A = \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} \]
\[ \lambda_1 = -1, \quad v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \lambda_2 = -2, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \Rightarrow \text{stable node} \]

@ \( (3,0) \):
\[ A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \]
\[ \lambda_1 = -3, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = -1, \quad v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \]

\[ \Rightarrow \text{stable node} \]

@ \( (1,1) \):
\[ A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \]
\[ \lambda_{1,2} = -1 \pm \sqrt{2}, \quad v_{1,2} = \begin{pmatrix} 1 + \sqrt{2} \\ 1 \\ -1 - \sqrt{2} \\ 1 \end{pmatrix} \]

\[ \Rightarrow \text{raddle} \]

c) Local phase portraits are:

It only matters what the trajectories do:

near the unstable node — how they go out at \( t \to -\infty \);

near the stable node — how they come in at \( t \to +\infty \).
2) Although there are separatrices here, they are found only by inspection. Namely:

One of the trajectories going out of (0,0) must come into (1,1);
Each of the two trajectories going out of (0,0) must come into (0,2) and (3,0).

This gives the following global phase portrait of (14), where the separatrices are shown by the thicker lines.

Interpretation:

1) The situations where either both species die out or coexist, are unstable. That is, any small deviation from these two equilibria will grow, i.e., the solution will not remain near (0,0) or (1,1).

2) The two stable equilibria are those where only one of the species remains and the other is extinct. Which species survives depends on the initial condition. Regions I and II of the phase space are the basin of attraction for (0,2), and III and V are the basin of attraction for (3,0).