Lecture 16: Qualitative solution of \( \dot{x} = f(x) \), where \( f(x) \) is a nonlinear function.

1. Introduction

In the rest of this course we will be using the notation \( \dot{x} \) (instead of \( x' \)) for \( dx/dt \), to be consistent with the notations of the book "Nonlinear dynamics & chaos" by S. H. Strogatz.

Our main goal of this part of the course is the qualitative solution of the system of two nonlinear equations:

\[
\begin{align*}
\dot{x} &= f(x, y) \\
\dot{y} &= g(x, y),
\end{align*}
\]

(1)

where \( f, g \) are real functions. However, in this lecture we will consider its baby model, a single equation

\[
\dot{x} = f(x),
\]

(2)

where \( f(x) \) may be a nonlinear function.

First, we will make two remarks about \( f(x) \).

1) \( x \) and \( f(x) \) are assumed to be real. For if not and \( x \) and \( f \) are complex, i.e.

\[
x = u + iv \quad \text{and} \quad f = \varphi(u,v) + i\psi(u,v),
\]

then (2) is actually a system of equations:

\[
\begin{align*}
\dot{u} + iv &= \varphi(u,v) + i\psi(u,v) \\
\dot{v} &= \varphi(u,v),
\end{align*}
\]

which is (1).
2) We also assume that \( f(x) \) does not explicitly depend on \( t \). Indeed, if not and \( f = f(x,t) \) (e.g., \( f = x + 2t \)), then (2) can be rewritten as
\[
\begin{align*}
\dot{x} &= f(x,t) \\
\dot{t} &= 1,
\end{align*}
\]
which, again, is a special case of (1).

**Def**: Equation (2) or system (1) where \( f(x) \), \( f(x,y) \), and \( g(x,y) \) do not explicitly depend on \( t \), are called \underline{autonomous}.

So, we will consider only \underline{autonomous} Eq. (2) (here) or Eqs. (1) (later).

2) **Qualitative solution of** \( \dot{x} = f(x) \)

One can often solve \( \dot{x} = f(x) \) analytically to find \( x(t) \). However, such solutions are rarely informative, unless one plots \( x \) vs. \( t \).

(see p. 16 in Strogatz).

Instead, one is more interested in the \underline{qualitative} behavior of \( x(t) \): does it grow, decay, oscillate, tend to a constant, etc.

**Def**: The solution \( x(t) \) is often called the \underline{trajectory}. 
Ex. 1 Consider \( \dot{x} = ax \). (3)

Let's plot the phase portrait for this system, which is \( \dot{x} \) vs \( x \):

\[
\begin{array}{c}
\begin{array}{l}
a > 0 \\
\begin{array}{c}
\dot{x} = ax \\
x > 0 \Rightarrow x \\
\dot{x} < 0 \Rightarrow x \\
x = ax
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{l}
a < 0 \\
\begin{array}{c}
\dot{x} = ax \\
x > 0 \Rightarrow x \\
\dot{x} < 0 \Rightarrow x \\
x = ax
\end{array}
\end{array}
\end{array}
\]

So the solution is unstable — it moves away from \( x = 0 \) on both sides of it.

So the solution is stable — it moves towards \( x = 0 \) from both sides.

Note that the above conclusions agree with the solution of (3), which is

\[
x = x_0 e^{at} \quad (4)
\]

\[
\begin{array}{c}
\begin{array}{l}
a > 0 \\
\begin{array}{c}
\dot{x} = ax \\
\text{moves away from } x = 0
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{l}
a < 0 \\
\begin{array}{c}
\dot{x} = ax \\
\text{moves towards } x = 0
\end{array}
\end{array}
\end{array}
\]

Def: If in (4), \( x_0 = 0 \), then \( x(t) = 0 \) for all times. The value \( x^* \) where \( f(x^*) = 0 \) is called the equilibrium, because
\[
\dot{x} = 0 \quad \text{at} \quad x = x_* . \quad (5)
\]

Another name of the equilibrium is the "fixed point" (used by Strogatz).

So, in Ex. 1, the trajectories move away from the unstable equilibrium when \( a > 0 \), and move towards the stable equilibrium when \( a < 0 \).

Ex. 2: Consider \( \dot{x} = f(x) \) where \( f(x) \) is shown below:

\[
\begin{align*}
\dot{x} > 0 & \Rightarrow x \uparrow \\
\dot{x} < 0 & \Rightarrow x \downarrow
\end{align*}
\]

All the trajectories move towards the stable equilibrium \( x_2 \) and away from the unstable equilibrium \( x_1 \).

Browse the Examples on pp 18-23 in Strogatz.
Ex. 3 Analyze the qualitative behavior of the solution of \( \dot{x} = \sin x \).

Will the solution oscillate?

NO !!

\[ x \]

\[ \dot{x} > 0 \]

\[ \dot{x} < 0 \]

The solution asymptotically approaches one of the stable equilibria but will never reach it or pass through it. Indeed, as we said in (5):

\[ \dot{x} = 0 \text{ at } x = x_* \text{ where } f(x_*) = 0, \quad (5) \text{ repeated} \]

which means that if the solution reaches the equilibrium, it remains there forever. But \( x_0 e^{-lt} \) never reaches \( x = 0 \), although it approaches it infinitely closely.

So, the following behavior is not possible:

Also, oscillating solutions are not possible in \( \dot{x} = f(x) \).
Linear stability analysis

Let $x$ be near an equilibrium point $x^*$. Then we can use the Taylor expansion:

$$f(x) = f(x^*) + f'(x^*)(x-x^*) + \ldots$$

so that (2) becomes:

$$\dot{x} = f(x^*) + f'(x^*)(x-x^*) + \ldots$$

or

$$(x-x^*)' = f'(x^*)(x-x^*) + \ldots \quad (6)$$

If we denote:

$$x-x^* = \Delta x, \quad f'(x^*) = a \quad (= \text{const, since } x^* \text{ is a fixed number})$$

then (6) becomes

$$\dot{\Delta x} = a \Delta x \quad (7)$$

which is (3). Thus, near an equilibrium:

$$\Delta x = x_0 e^{at}, \text{ or } x = x^* + (x_0 - x^*) e^{at}$$

$f'(x^*) > 0 \Rightarrow \text{unstable equilibrium}$

$f'(x^*) < 0 \Rightarrow \text{stable equilibrium}$
4) Special cases where the linear stability analysis fails.

When \( f'(x^*) = 0 \), we have:

\[
(X - X^*)' = \frac{f''(x^*)}{2} (X - X^*)^2 + \frac{f'''(x^*)}{6} (X - X^*)^3 + \ldots
\]

So we may have:

1) \( b = \frac{f''(x^*) \neq 0}{=} \quad \Rightarrow \quad \dot{X} = bX^2 \quad (8) \)

When \( b > 0 \):

we have a \underline{remitable equilibrium}
(and similarly for \( b < 0 \)).

2) \( b = 0, \ c = f'''(x^*) \neq 0 \quad \Rightarrow \quad \dot{X} = cX^3 \quad (9) \)

\[\begin{array}{ll}
\text{Unstable} & \text{Stable} \\
\end{array} \]

At home you will show that the behavior near such an equilibrium is \underline{slower than exponential}.\]
5. **Bifurcations**

Suppose \( \dot{x} = f(x, r) \), where \( r \) is some coefficient (parameter). E.g., \( \dot{x} = r + x^2 \). As we change \( r \), \( f(x, r) \) changes. As a result:

- the number of equilibria, and/or
- their stability/instability can change. Such changes are called **bifurcations**. The values of \( r \) where bifurcations occur are called **bifurcation points**.

**A. Saddle-node bifurcation** (reason for this name - later)

"Normal form" of the equation for this bifurcation is

\[
\dot{x} = r + x^2
\]  --- (10)

\( r < 0 \)

\[
\begin{align*}
\dot{x} & \quad \text{one stable + one unstable equilibria} \\
& \quad \text{senustable} \\
& \quad \text{no equilibria}
\end{align*}
\]

**Bifurcation diagram:**

\[
\begin{align*}
& \quad \text{unstable} \\
& \quad \text{no equilibria} \\
& \quad \text{stable}
\end{align*}
\]
Alternative forms of the normal form (10) of the saddle-node bifurcation are:

\[ \dot{x} = r - x^2 \]  \hspace{1cm} (11a)
\[ \dot{x} = -r + x^2 \]  \hspace{1cm} (11b)

Note: Referring to (10), i.e., \( \dot{x} = r + x^2 \), suppose \( r > 0 \), so that there is no equilibrium (the particle is always in motion).

But near the point \( x = 0 \), the motion slows down because \( \dot{x} \) becomes smaller.

This slowing down is a signature of a "nearby" saddle-node bifurcation.

Here is another variation of the saddle-node bifurcation; consider a cubic curve.

Note that on a vertical line \( r = \text{const} \), the stable and unstable equilibria alternate. This is a generic property for any bifurcation.
B Transcritical bifurcation

Normal form: \( \dot{x} = rx - x^2 \) (12)

\( r < 0 \)

\( \dot{x} \)

\( r = 0 \)

\( \dot{x} \)

\( r > 0 \)

Bifurcation diagram:

```
           \( X^* \)
    \( \sim \) stable

stable \( - \) unstable \( r \)

unstable
```

"The zero and nonzero equilibria exchange their stability."

C Pitchfork bifurcation

C(i) Supercritical pitchfork bifurcation

Normal form: \( \dot{x} = rx - x^3 \) (13)

Bifurcation diagram:

\( x^* = \sqrt{r} \) stable

\( r \dot{x} = x^* \)

\( r \dot{x} = -x^* \)

\( x^* - x^2 = 0 \Rightarrow x^* (r - x^2) = 0 \Rightarrow x^* = 0 \& \ x^* = \pm \sqrt{r} \quad (r > 0) \)
Ex. 4 | Draw a bifurcation diagram for
\[ x' = -x + \beta \cdot \tanh x, \quad \beta > 0 \]

First, we will draw each of the two terms of \( f(x) \) and from there conclude about the shape of \( f(x) \).

\[ \beta < 1 \]

Now draw the phase portraits:

\[ \beta > 1 \]

Near \( \beta = 1 \) we have a supercritical pitchfork bifurcation; so \( \beta = 1 \) is the bifurcation point.
C(ii) Subcritical bifurcation

Normal form: \[ \dot{x} = rx + x^3 \] (15)

At home you will show that the bifurcation diagram here is:

\[ \text{unstable} \quad \text{stable} \quad \text{unstable} \quad r \]

Note that, as before, along a vertical line \( r = \text{const} \), the stability of the equilibria alternates between "stable" and "unstable".

D A more complex version of the subcritical pitchfork bifurcation — hysteresis.

Normal form: \[ \dot{x} = rx + x^3 - x^5 \] (16)

Bifurcation diagram:
Hysteresis:

As we slowly increase \( r \), the particle will follow the equilibrium as shown.

If we very slowly change \( r \), we can imagine that the particle, initially sitting at a stable equilibrium \( x^*(r) \), will continue to sit there until that equilibrium becomes unstable (due to the change in \( r \)).

If we slowly decrease \( r \), the particle will follow the equilibrium in a different way.

Thus, for \( 0 < r < r_0 \), whether the particle will sit at the zero or nonzero equilibrium will depend on the history of how \( r \) was changed.

Often, one draws just one picture:

\[
\begin{array}{c}
\text{hysteresis area} \\
\end{array}
\]
Asymmetric pitchfork bifurcation

Normal form: \[ \dot{x} = h + rx - x^3, \quad h > 0 \]

\[ (17) \]

Question: What other earlier considered bifurcation does this look like?