1. Background and motivational examples.

(1) Background on complex numbers

- Complex number: \( z = x + iy \) (\( x, y \in \mathbb{R} \))
  \[
  z = \sqrt{x^2 + y^2} \left( \frac{x}{\sqrt{x^2 + y^2}} + i \frac{y}{\sqrt{x^2 + y^2}} \right)
  = |z| \cdot (\cos \phi + i \sin \phi)
  \]
  \( |z| = \sqrt{x^2 + y^2} \) → modulus (absolute value) of \( z \)
  \( \phi = \arctan \left( \frac{y}{x} \right) + \pi \) if \( x < 0 \) → argument of \( z \)
  \( \arg(z) \).

- Euler formula (verified by Maclaurin series):
  \[
  \cos \phi + i \sin \phi = e^{i\phi}
  \]

- Corollary: \( z = |z|e^{i\phi} \) (polar form of a complex number).

- For any real \( \phi \):
  \[
  |e^{i\phi}| = |\cos \phi + i \sin \phi| = \sqrt{\cos^2 \phi + \sin^2 \phi} = 1.
  \]
  \[
  |e^{i\phi}| = 1 \quad (\phi \in \mathbb{R}).
  \]

- If \( z \) and \( w \) are two complex numbers, then:
  \[
  |zw| = |z| |w|, \quad \frac{|z|}{|w|} = |z|/|w|.
  \]
  \[
  \arg(zw) = \arg(z) + \arg(w); \quad \arg(z) = \arg(z) - \arg(w).
  \]

Note: There are **no** similar rules for \((z \pm w)\)!
Corollary:
\[ |e^{x+iy}| = |e^x \cdot e^{iy}| = |e^x| \cdot |e^{iy}| = e^x \]
\((x, y \in \mathbb{R}).\)

- Complex conjugate.
  If \( z = x + iy \), then \( \overline{z} = x - iy \) is called the complex conjugate of \( z \).

  Properties:
  1) \( z \cdot \overline{z} = x^2 + y^2 = |z|^2 \)
  2) \( \frac{\overline{z}}{z} = \frac{1}{e^{i\phi}} \Rightarrow \overline{z} = |z| \cdot e^{-i\phi} \)
  3) \( z + \overline{z} = (x + iy) + (x - iy) = 2x = 2 \cdot \text{Re}(z) \)
     \( z - \overline{z} = 2i \cdot y = 2i \cdot \text{Im}(z) \).

  In particular:
  \[
  e^{i\Phi} + e^{-i\Phi} = 2 \cos \Phi \\
  e^{i\Phi} - e^{-i\Phi} = 2i \cdot \sin \Phi.
  \]

- Notation: Often, instead of \( z + \overline{z} \), one writes "\( z + \text{c.c.} \)". Thus,

  \[
  \cos \Phi = \frac{1}{2} (e^{i\Phi} + \text{c.c.}) \\
  \sin \Phi = \frac{1}{2i} (e^{i\Phi} - \text{c.c.})
  \]

  Also, \( z + \text{c.c.} = \text{real} \# \)
  \( z - \text{c.c.} = \text{imaginary} \# \).

Note: We'll prove later that any linear combination of \( \sin \) & \( \cos \) can be written as some other lin. comb. of \( e^{ix} \) and \( e^{-ix} \).
4) If \( \omega = |\omega| e^{i\psi} \), then
\[
|\omega| e^{i\phi} + \text{c.c.} = |\omega| e^{i\phi} + |\omega| e^{-i\psi}
\]
\[
= 2 |\omega| \cos (\phi + \psi).
\]

1 Linear systems in electrical networks.

Example 1

Ohm's law: The voltage drop \( V \) across a resistor is the product of the current and the resistance:
\[ V = I \cdot R. \]

Kirchhoff's 1st law: At any node:
\[ \sum \text{in} I = \sum \text{out} I \]

Kirchhoff's 2nd law: The algebraic sum of voltage drops and voltage sources (batteries) around a closed loop is zero.

Solution: Let us assign some direction to the currents in each branch. If these directions turn out to be wrong, this will be indicated by the negative value of the current.

\[ K \text{ 1st law @ A: } I_1 = I_2 + I_3 \quad (1) \]

\[ K \text{ 1st law @ B: } I_2 + I_3 = I_1 \quad \text{(same as (1))} \]
K 2nd Law/top loop (counterclockwise):
\[ E_1 - I_1R_1 + I_3R_3 = 0 \] (2)
K 2nd law/bottom loop
\[ E_1 - I_3R_3 + I_2R_2 + E_2 = 0 \] (3)
K 2nd outer loop
\[ E_1 - I_1R_1 + I_2R_2 + E_2 = 0 \]

Thus:
\[ I_1 - I_2 - I_3 = 0 \] (1)
\[ R_1I_1 - R_3I_3 = E_1 \] (2)
\[ -R_2I_2 + R_3I_3 = E_2 \] (3)

\[
\begin{pmatrix}
1 & -1 & -1 \\
R_1 & 0 & -R_3 \\
0 & -R_2 & R_3 \\
\end{pmatrix}
\begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
E_1 \\
E_2 \\
\end{pmatrix}
\]

\[ A \cdot \mathbf{x} = \mathbf{b} \]

Solve this for the unknown \( \mathbf{x} \) to find the currents.

Q1: What properties of matrix \( A \) guarantee that a unique solution exists?

Q2: If we change the \( R \)'s and \( E \)'s slightly, can we be sure that the \( I \)'s will also change only slightly?

Browse Sec. 2.4 for other examples of networks.

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Example 2

\[ \text{Find the equation for the current in this time-dependent circuit.} \]
Solution. In addition to the Ohm's law, we also need:

- Voltage drop across capacitor: \( V_C = \frac{q}{C} \), \( q = \text{charge across capacitor} \), \( C = \text{capacitance} \).
- Volt drop across inductance: \( V_L = L \frac{dI}{dt} \), \( L = \text{inductance} \).

Relation between \( q \) and \( I \): \( \frac{dq}{dt} = I \).

Then, 2nd law:
\[
E(t) - V_R - V_C - V_L = 0.
\]

\[
E(t) = I \cdot R + \frac{q}{C} + L \frac{dI}{dt},
\]
\[I = \frac{dq}{dt}.
\]

Let \( x = \begin{pmatrix} q \\ I \end{pmatrix} \). Then the above system is:
\[
\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ \frac{1}{L} & -\frac{R}{L} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{E(t)}{L} \end{pmatrix}.
\]

Let us denote \( \frac{dx}{dt} = x'(t) \) or \( \dot{x}(t) \).
Then the above system has the form
\[
\dot{x}(t) = Ax + g(t), \quad (\text{A})
\]

Q3: How to find the most general solution of this equation? (See part 2 of this course.)
Q4: W/o solving for \( x(t) \), can we predict its general properties (growing/oscillatory, decaying if goes to constant) if we know \( A \) and \( u(t) \)?

Remark about notations. Note that out of the two components of \( x(t) = (\varphi, \psi) \), we want only \( \psi \). Then, if we denote \( \psi(t) = y \), we can restate (\( \star \)) as follows:

\[
\begin{align*}
\dot{x} &= Ax + u(t) \\
y &= (0 \ 1)x + 0 \cdot u(t),
\end{align*}
\]

or in general form:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

(25T)

In the notations of Control Systems Theory,
- \( x \) = state variable
- \( u \) = control variable
- \( y \) = output variable.

Prof. Mirchandani will give a guest lecture providing more specific examples.

(2) Masses on springs. (See 2.5)

Example 3 Determine the motion of the two masses \( m_1, m_2 \).

[Diagram of two masses on springs with forces and positions labeled]

Netwon's 2nd law:

\[
m\ddot{x} = \sum F, \quad \ddot{x} = \ddot{x}(t)
\]
Hook's law: \( F = -K \cdot \Delta x \) (change in spring's length).

Solution:

\[
\begin{align*}
\text{mass 1: } & \quad m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2) \\
\text{mass 2: } & \quad m_2 \ddot{x}_2 = -k_2(x_2 - x_1)
\end{align*}
\]

\[
\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

\[
M \cdot \ddot{x} = -K \cdot x
\]

Thus, we need to solve \( M \ddot{x} = -K \ddot{x} \).

We expect oscillations, so we look for \( x = e^{i\omega t} \).

\[
\begin{align*}
\text{because } & \quad e^{i\omega t} \text{ is a linear combination of } \\
& \quad \cos(\omega t) \text{ and } \sin(\omega t) \text{(see Section 02)}
\end{align*}
\]

\[
\begin{align*}
M \cdot (e^{i\omega t})'' &= -K \cdot (e^{i\omega t})' \\
M \cdot (e^{i\omega t})'' &= -K \cdot e^{i\omega t} \\
(K - \omega^2 M) \cdot (e^{i\omega t}) &= 0.
\end{align*}
\]

This obviously holds for \( \omega = 0 \), but we want \( \omega \neq 0 \).

From undergraduate L.A., \( A \cdot x = 0 \) has a solution \( x \neq 0 \) iff \( A \) is singular. Thus we need to find such \( \omega^2 \) that \( (K - \omega^2 M) \) is singular.

A convenient tool to decide: \( \det(K - \omega^2 M) = 0 \).

\[
\begin{align*}
\det \begin{vmatrix} (k_1 + k_2) - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix} = 0
\end{align*}
\]
\((k_1 + k_2 - \omega^2 m_1)(k_2 - m_2 \omega^2) - (-k_2)(-k_2) = 0\)

\((\omega^2)^2 m_1 m_2 - (\omega^2) \cdot (k_1 + k_2) m_2 + k_1 k_2 = 0\).

At this point, let's use the numbers suggested by the book: \(k_1 = 80 \text{ N/m}, k_2 = 40 \text{ N/m}, m_1 = 10 \text{ kg}, m_2 = 5 \text{ kg}\).

Then (see book): \(\omega_1 = 2, \omega_2 = -2, \omega_3 = 4, \omega_4 = -4\).

Next, what \(v_0(\theta)\) corresponds to each of these \(\omega\)'s?

\(\omega_1 = 2\)

\[
\begin{pmatrix}
120 & -2^2 & 10 & -40 \\
-40 & 40 & -5^2 & 2^2 \\
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\end{pmatrix}.
\]

\[80 \alpha - 40 \beta = 0 \implies 2\alpha = \beta, \quad \text{not independent.}
\]

\[-40 \alpha + 20 \beta = 0 \quad \text{(Note: 2 eqns. are dependent - this is because \((K - \omega^2 M)\) is singular.)}
\]

\[\implies 2\alpha = \beta, \quad \Rightarrow \quad v_1 = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \alpha_2, \quad \alpha_2 \text{ arbitrary.}
\]

\(\omega_2 = -2\)

Same: \(\omega_2 = \omega_1 = \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \alpha_2\)

\(\omega_3 = 4\)

\(v_3 = \begin{pmatrix} \alpha_3 \\ -1 \end{pmatrix} \alpha_3, \quad \alpha_3 = \alpha_3 e^{4it}
\]

\(\omega_4 = -4\)

\(v_4 = \begin{pmatrix} \alpha_4 \\ -1 \end{pmatrix} \alpha_4, \quad \alpha_4 = \alpha_4 e^{4it}
\]

Thus:

\[
\chi = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2it} + \alpha_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2it} + \alpha_3 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4it} + \alpha_4 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-4it}
\]

Since \(\chi\) must be real,

\[
\chi = \begin{pmatrix} 1 \\ 2 \end{pmatrix} (\alpha_1 e^{2it} + c.c.) + \begin{pmatrix} -1 \\ 1 \end{pmatrix} (\alpha_3 e^{4it} + c.c.)
\]

If \(\alpha_1 = |\alpha_1| e^{i\theta_1}, \quad \alpha_3 = |\alpha_3| e^{i\theta_3}, \quad \text{then (see Sec. 2)}\)
\[(x_1) = (\frac{1}{2})x_1 \cos(2t + \psi_1) + (-\frac{1}{2})x_3 \cos(4t + \psi_3).\]

I.e. the displacements of the masses are lin. superpositions of two sinusoids with frequencies 2 and 4.

Observation: The eigenvalues \( \lambda \) played a prominent role in the solution.

Q5: Can we be sure that in other similar problems, the eigenvalues \( \lambda \) are real-valued?

Q6: For any \( \omega^2 \) that makes \((K-\omega^2M)\) singular, there is always one \( \psi \) s.t. \((K-\omega^2M)\psi = 0\). But if \((\omega^2)\) is a repeated root (e.g., if \((\omega^2) = (\omega^3) = 2^2\)), then we would require two different \( \psi \)'s that satisfy \((K-(\omega^2)M)\psi = 0\). So, questions:

(a) What does it mean "two different \( \psi \)'s"?
(b) And do these two different \( \psi \)'s always exist? If they don't (i.e., if there is only one distinct \( \psi \)), how does this affect the solution \( x(t) \)?

3. Population models
Example 4 (Sec. 2.3 modified)
Let there be two populations, of foxes and chickens, whose numbers in the 1st year are \( F^{(1)} \) and \( C^{(1)} \). W/o foxes eating chickens, the latter would multiply so that every year their number grows by 20%:
\[
C^{(2)} = 1.2 \ C^{(1)}, \quad \text{etc.}
\]
W/o chickens, foxes would die out:
\[
F^{(2)} = 0.6 \ F^{(1)}, \quad \text{etc.}
\]
But since foxes eat chickens at rate \( K \), the above eqs. are modified:
\[
C^{(2)} = 1.2 \ C^{(1)} - K \ F^{(1)}
\]
\[
F^{(2)} = 0.5 \ C^{(1)} + 0.6 \ F^{(1)}
\]
Assumed increase of the foxes population due to eating chickens.

If these trends are preserved, what happens to the two populations over many years?

Solution
As in Ex. 3, the answer is found using eigenvalues and eigenvectors,
1) Let \( \mathbf{x} = \begin{pmatrix} C \\ F \end{pmatrix} \). Then
\[
\mathbf{x}^{(2)} = \begin{pmatrix} 1.2 & -K \\ 0.5 & 0.6 \end{pmatrix} \mathbf{x}^{(1)}
\]
\[
\mathbf{x}^{(3)} = \mathbf{A} \mathbf{x}^{(2)} = \mathbf{A} (\mathbf{A} \mathbf{x}^{(1)}) = \mathbf{A}^2 \mathbf{x}^{(1)}
\]
\[
\mathbf{x}^{(m+1)} = \mathbf{A}^m \mathbf{x}^{(1)} = \mathbf{A}^m \mathbf{x}^{(1)}
\]
\[
\mathbf{x}^{(n+1)} = \mathbf{A}^n \mathbf{x}^{(1)} = \mathbf{A}^n \mathbf{x}^{(1)}
\]
2) Let $v_1$ and $v_2$ be two eigenvectors of $A$ with eigenvalues $\lambda_1, \lambda_2$:

\[ A v_k = \lambda_k v_k, \quad k = 1, 2. \]

Note: Here we have assumed that these two different eigenvectors exist. We can explain what "different" here means because $v_1$ and $v_2$ are just vectors in the $\mathbb{R}^2$ plane: different $\neq$ non-parallel.

When $v_1, v_2$ are different, any initial vector $x^{(0)}$ is a superposition of $v_1$ and $v_2$:

\[ x^{(0)} = c_1 v_1 + c_2 v_2 \]

Then

\[ x^{(2)} = A x^{(1)} = A (c_1 v_1 + c_2 v_2) = c_1 A v_1 + c_2 A v_2 = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2. \]

\[ x^{(3)} = A x^{(2)} = c_1 \lambda_1 A v_1 + c_2 \lambda_2 A v_2 = c_1 \lambda_1^2 v_1 + c_2 \lambda_2^2 v_2. \]

\[ x^{(n+1)} = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2. \]

Thus, if at least one $\lambda > 1$, then since $\lambda^n \to \infty$, the populations will grow (see Example 2.16 in book). If both $\lambda$'s $< 1$, then both
Q8: Compare Examples 3 and 4.

Both involved eigenvalues (\(\omega^2\) in Ex. 3 and \(\lambda\) in Ex. 4) and eigenvectors. Based on our intuition, we expect that the coupled springs system in Ex. 3 will always perform oscillations irrespective of what numbers we choose for \(k_{1,2}\) and \(m_{1,2}\). Oscillations mean that \(\omega^2 > 0\). This must somehow follow from the mathematical structure of matrix \((K - \omega^2 M)\).

On the other hand, you will see in a new problem that \(\lambda\)'s in Ex. 4 can be positive, negative, or complex.

So, is there something in the general form of the matrix that could tell us whether its eigenvalues should always be real? Or always positive?
populations will die out (Example 2.17 in book).

Q7: Are there other scenarios? i.e., what if both $Xs = 1$? Or what if they are complex? And can we always find two non-parallel $v_1, v_2$? (Same as Q6(b)).

Q8: How sensitive is the evolution scenario to small changes of the entries of matrix $A$? (Similar to Q2. See also a HW problem. Browse Sec. 2.2 for other scenarios of evolution.)

Remark. The eigenvalues/eigenvectors played an important role here because the equation was linear: $A (c_1 v_1 + c_2 v_2) = c_1 A v_1 + c_2 A v_2$. 

There are nonlinear systems where this is not true. An example is below.

Example 5. (More realistic population model). The act of eating of a chicken by a fox occurs when the two meet. The probability of such a meeting is proportional to the densities of populations of both species. Then the eqs in the previous example are replaced with:

$C^{(2)} = 1.2 C^{(1)} - K F^{(1)} C^{(0)}$

$F^{(2)} = 0.5 C^{(0)} F^{(0)} + 0.8 F^{(1)}$.

The terms $C^{(0)} F^{(0)}$ make this system nonlinear, and the linear superposition
principle (**) no longer holds.

Q10: How can we employ the techniques of linear systems analysis for analyzing nonlinear systems?

The 3rd part of this course will be devoted to answering this question.