**Instructions**: Present your work in a neat and organized manner. Please use either the 8.5 × 11 size paper or the filler paper with pre-punched holes. Please do not use paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but not by folding its (upper left) corner.

You must show all of the essential details of your work to get full credit. If I am forced to fill in gaps in your solution by using nontrivial (at my discretion) steps, I will reduce your score for that particular assignment.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting your own work.

All problems in a given assignment contribute equal amount to the assignment’s total score, unless otherwise noted.

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**Homework Assignment #9**

**Due Friday, October 31, 2008**

1. (a) Consider the \( N \times N \) matrix \( F_N \), introduced in Section 6 of Lecture 3:

\[
F_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1, e^{it}, \ldots, e^{i(N-1)t} \end{bmatrix},
\]

where vectors \( e^{ikt} \) were defined in Problem 2(b) of HW 4. Use the result of that problem to show that \( F_N \) is a unitary matrix.

(b) Let the discrete Fourier transform \( \hat{\mathbf{x}} \) of a discrete signal \( \mathbf{x} = [x_0, x_1, \ldots, x_{N-1}]^T \) be defined by the relation \( \mathbf{x} = F_N \hat{\mathbf{x}} \). (The usual definition of the discrete Fourier transform differs from this definition by the factor \( \sqrt{N} \).) Use the result of part (a) to express \( \hat{\mathbf{x}} \) in terms of \( \mathbf{x} \); that is, write the \( k \)th entry \( \hat{x}_k \) of vector \( \hat{\mathbf{x}} \) as a linear combination of entries of vector \( \mathbf{x} \). (To verify your answer, find a similar result in Lecture 6.)

(c) Use the result of part (a) and Theorem 1 of Lecture 11 to prove the Parceval identity:

\[
\| \mathbf{x} \|_2^2 = \| \hat{\mathbf{x}} \|_2^2,
\]

where the norm is defined with respect to the standard inner product in \( \mathbb{C}^N \).

*Note 1*: This provides a proof for the results observed in Examples 5 and 6 of Lecture 4.

*Note 2*: You may be more familiar with the continuous version of the Parceval identity:

\[
\int |x(t)|^2 dt = \int |\hat{x}(\omega)|^2 d\omega,
\]

where \( \hat{x}(\omega) \) is the Fourier transform of the continuos signal \( x(t) \).

(d) Without doing any calculations, but instead referring to earlier homework problems and to the proof of Theorem 1(iv), state and explain whether a counterpart of the Parceval identity will hold for the Laplace transform of \( x(t) \).

2. This problem is worth 0.5 of a regular problem. The parts in this problem are not related to each other.

(a) Find numbers \( a, b, c \) such that matrix \( Q \) is orthogonal, where

\[
Q = \begin{pmatrix}
-3/\sqrt{35} & 1/\sqrt{6} & a \\
1/\sqrt{35} & -2/\sqrt{6} & b \\
5/\sqrt{35} & 1/\sqrt{6} & c 
\end{pmatrix}.
\]

You are encouraged to use Matlab for auxiliary calculations. Attach your printouts or state which results you found with Matlab.
(b) Use results of Section 1 of Lecture 11 to prove that the condition number of any unitary matrix is 1, where the condition number is defined with respect to the 2-norm in \( \mathbb{C}^N \).

Note: Since, as you proved in Problem 1 of HW 7, the condition number with respect to any subordinate norm is never less than 1, then one can say that unitary matrices are the most well-conditioned matrices (and hence are the farthest matrices from singular ones). This is intimately related to the fact that to find the inverse of a unitary matrix, one does not need to do any calculations.

3. Sec. 8.2, # 4. Please follow the lines of Example (8.4) and the directions listed below. Let us denote \( B \overset{\text{def}}{=} A^T \).

You are encouraged to use Matlab to multiply matrices and find eigenvectors. Unlike in Example (8.4), you may present your answer in decimal form with 4 significant figures. As usual, you must attach your printout or indicate which results you obtained with Matlab.

Useful commands for this problem are those allowing one to select only certain rows or columns in a matrix. E.g., \( B(1,:) \) selects the first row of \( B \), and \( B2=B(2:3,2:3) \) defines a \( 2 \times 2 \) block \( B2 \) to be made by the entries of \( B \) in the 2nd and 3rd rows and 2nd and 3rd columns. Similarly, \( C(2:3,2:3)=D \), where \( D \) is a \( 2 \times 2 \) matrix, defines a new matrix \( C \) whose lower-right \( 2 \times 2 \) block is set to equal \( D \); the other entries of \( C \) can then be defined separately. Those entries that were not defined are defaulted to zero. For more details, see any of Matlab tutorials posted on the course web page.

- As in Example (8.4), start with \( \lambda_1 = 0.6 \) and the corresponding eigenvector, that equals the first column of matrix \( Q \) in Problem 2(a) above.
- You must exhibit the \( 3 \times 3 \) orthogonal matrix by which you make the similarity transformation from the matrix (counterpart of \( A_1 \) in Example (8.4)) having the form

\[
B_1 = \begin{pmatrix}
0.6 & * & * \\
0 & * & * \\
0 & * & *
\end{pmatrix}
\]

\[
to the matrix having the final required form
\[
T = \begin{pmatrix}
0.6 & * & * \\
0 & * & * \\
0 & 0 & *
\end{pmatrix}
\]

- You must explain why the similarity transformation from the original matrix \( B \) to the final upper-triangular matrix \( T \) can be viewed as performed using a single orthogonal matrix, as predicted by Theorem 3 of Lecture 11. (This is the gap in the proof of that theorem that I mentioned in class and promised that you would fill in at home.) Hint: Use a result of Theorem 1 in Lecture 11.

Unlike in Example (8.4), you must exhibit not only the upper-triangular matrix \( T \), but also the orthogonal matrix \( Q \) referred to in Theorem 3.

Note: It is a good idea to verify your answer with Matlab. Another sanity check is to realize what entries should be on the main diagonal of \( T \) (recall Problem 1(i) of HW 8).

4. This problem is worth 0.75 of a regular problem.

(i) Sec. 8.3, # 6.

Hint: You can do part (b) most easily using the results of (a) and of Problem 1 in HW 8.

(ii) Show that any matrix \( C \) can be written as \( C = A + iB \), where \( A \) and \( B \) are both some Hermitian matrices.

Hint: Consider \( C + C^H \) and \( C - C^H \). The result of part (i)(a) may also help.
5. This problem is worth 0.5 of a regular problem.

Sec. 8.3, # 4.

Note: Please do not look at the answer at the back of the book (at least before you finished the problem). The point of this problem is to make you find an orthonormal set of eigenvectors even though you may originally come up with a non-orthogonal set. See the long Note 1 after the Corollary to Theorem 4 in Lecture 11, and/or Example (8.11) in the textbook and the paragraph before it. As you construct a new orthonormal set out of the original non-orthogonal one, you must explain why the new orthogonal vector(s) that you find is(are) still the eigenvector(s) of the matrix in question.

6. This problem is worth 1.25 regular problems.

This problem refers to the setup of # 4 of Sec. 2.5; see the figure at the bottom of p. 65 and the system at the top of p. 66. You already considered part (a) of that problem in HW 1. In this assignment, assume that all masses are equal: $M = m$. As previously, let us denote the matrix containing the spring constant as $K$.

Recall that in Example 3 of Lecture 1, we first used the substitution $x(t) = e^{i\omega t}v$ to derive the eigenvalue problem whose eigenvalue and eigenvector were $\omega^2$ and $v$, and then took a linear combination of all such possible solutions. Here we will use an equivalent, but more literate approach.

You may use Matlab or Mathematica. Relevant Mathematica commands can be gleaned from the notebook that I posted for Lecture 10.

(a) Without any calculation, but referring only to results from Lecture 11, explain why the eigenvectors, $v_i$, of $K$ are guaranteed to form a basis in $\mathbb{R}^p$, where you also need to answer what $p$ is. (Note that you did not know how to explain that in HW 1.) Was the condition $M = m$ required for this conclusion? Finally, find all eigenvalues and eigenvectors explicitly for this problem.

(b) Look for the solution $x(t) = [X_1, X_2, X_3]^T$ in the form

$$x(t) = c_1(t) v_1 + \cdots + c_p(t) v_p,$$  \hspace{1cm} (HW9.5.1)

where $v_1, \ldots, v_p$ are the eigenvectors of $K$. Substitute (HW9.5.1) into the linear system displayed at the top of p. 66. Use the method of Theorem 2 in Lecture 6 (Theorem (5.74) of the book) to derive a system of decoupled ODEs for $c_i(t)$:

$$c_i''(t) = \cdots, \quad i = 1, \ldots, p, \hspace{1cm} (HW9.5.2)$$

where the prime denotes $d/dt$. You must explain what property of the eigenvectors $v_i$ allows you to use this method. (See Lecture 11.) How do the eigenvalues of $K$ enter into system (HW9.5.2)?

Note: While the equations for the displacements $X_1, X_2, X_3$ are coupled to one another, the equations for the coefficients $c_1, \ldots, c_p$ are decoupled. Therefore, they are much easier to solve than the original coupled system. This method of expanding the general solution of a linear system into decoupled modes, as in (HW9.5.1), is very widely used in sciences, engineering, and mathematics. Thus, you have seen another interpretation of eigenvectors of a matrix: they represent independent modes of a linear system.

(c) Suppose the linear system at the top of p. 66 is given initial conditions:

$$x(0) = [\alpha_1, \alpha_2, \alpha_3]^T, \quad \frac{dx}{dt}(0) = [\beta_1, \beta_2, \beta_3]^T.$$  \hspace{1cm} (HW9.5.3)

Use (HW9.5.1) and (HW9.5.3) to derive initial conditions for $c_1, \ldots, c_p$ and $c_1', \ldots, c_p'$.

You do not need to use the explicit form of the eigenvalues and eigenvectors, but can leave the answer in the general form.
(d) Now, assume that in part (c), $\beta_1 = \beta_2 = \beta_3 = 0$, and use the explicit form of the eigenvectors $v_i$ that you found in part (a) to obtain the initial conditions for $c_1, \ldots, c_p$ and $c'_1, \ldots, c'_p$. Use these initial conditions to solve of the decoupled system (HW9.5.2) for the mode amplitudes $c_i(t)$.

Note that this solution along with (HW9.5.1) will satisfy the original coupled system of equations for $X_1, X_2, X_3$.

(e) You probably noticed that one of the eigenvalues that you found in part (a) is zero. This means that there is a mode that is not oscillatory but instead corresponds to pure translation of the mass–spring system as a whole. This fact becomes evident if you look at the figure at the bottom of p. 65. You may also see it by examining the form of the eigenvector corresponding to the zero eigenvalue.

Now verify that this purely translational mode of motion is destroyed and is replaced by an oscillatory mode if we connect one of the end masses, say the left mass, to a wall with a spring. Namely, derive the corresponding modified system for $X_1, X_2, X_3$ and then calculate the eigenvalues of the modified matrix $K$. Use the same constant $k$ for the new spring as for the other springs. (You do not need to compute eigenvectors or repeat any of the work you did in parts (a)–(d).)