Instructions: Present your work in a neat and organized manner. Please use either the 8.5 × 11 size paper or the filler paper with pre-punched holes. Please do not use paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but not by folding its (upper left) corner.

You must show all of the essential details of your work to get full credit. If I am forced to fill in gaps in your solution by using not trivial (at my discretion) steps, I will reduce your score for that particular assignment.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting your own work.

All problems in a given assignment contribute equal amount to the assignment’s total score, unless otherwise noted.

Homework Assignment # 8
Due Monday, October 27, 2008

1. This problem is easy, although somewhat time-consuming.
   
   (i) Sec. 7.2, # 5.
   
   (ii) Sec. 7.2, # 11(b–e).
   
   Hint: For all these problems, start with \( AV = \lambda V \). For (c), use induction (or, equivalently, exhibit a convincing pattern). For (d), act with \( A^{-1} \) on both sides of \( AV = \lambda V \).
   
   (iii) If \( v_1, v_2 \) are two eigenvectors of \( A \) corresponding to the same \( \lambda \), prove that \( c_1 v_1 + c_2 v_2 \) is also an eigenvector with the same \( \lambda \).

2. This problem is worth 0.5 of a regular problem.

   How many eigenvectors can you find (by hand) for each of these matrices:

   \[
   A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}
   \]

   Is either of these matrices diagonalizable? Please explain.

   Food for thought: (You are advised, but not required, to do the following.) Check your answers with Matlab. Type `help eig` for the proper syntax of the command. Do Matlab’s answers agree with yours? What general (i.e., not mathematics-related) conclusion can you draw from this? Again, you are not required to write down your answer. This is just a piece of food for thought.

3. For \( A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \), find (by hand) its only eigenvector, \( v \), and normalize it so that \( ||v|| = 1 \). Obviously, \( A \) cannot be diagonalized.

   Now, find vector \( u \) that satisfies three conditions:

   • \( (A - \lambda I)u = c v \), where \( \lambda \) is the eigenvalue of \( A \) and \( I \) is the \( 2 \times 2 \) identity matrix;
   • \( u \) is chosen to be orthogonal to \( v \) (by Theorem 2 of Lecture 2, the above equation for \( u \) has infinitely many solutions that differ by a multiple of \( v \); out of them you can find only one solution that is orthogonal to \( v \));
   • the constant \( c \) is adjusted so that \( ||u|| = 1 \). (I.e., first solve the equation for the general \( c \) and then adjust it.)
Finally, follow the steps of the derivation of Eqs. (10) and (11) in Lecture 9 to derive the following decomposition of $A$: $A = Q T Q^{-1}$. Exhibit $Q$, $Q^{-1}$, and $T$ explicitly. 

Describe (briefly) the form of $T$ and how $Q^{-1}$ seems to be related to $Q$.

**Note:** The situation that you observed in this problem turns out to be the general situation for non-diagonalizable matrices, as we will show in a later lecture.

4. **This problem is worth 0.5 of a regular problem.**

   (a) Prove that if a matrix is diagonalizable and if all of its eigenvalues are zero, than this matrix is the zero matrix.

   (b) Find the eigenvalues and eigenvector(s) of $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. From this example, write a conclusion related to the statement in part (a).

5. **This problem is worth 0.5 of a regular problem.**

   Given the definition of $e^{At}$:
   
   $e^{At} \overset{\text{def}}{=} I + tA + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \cdots,$

   where $A$ is a constant matrix, verify that $X = e^{At}$ satisfies $dX/dt = AX$.

   **Hint:** To verify (as opposed to solve) an equation, you need to substitute the expression in question into both sides and confirm that they are equal.

6. **This problem is worth 0.75 of a regular problem.**

   (a) Let $A$ and $B$ both be diagonalizable and nonsingular matrices. Prove the following interesting result: the eigenvalues of $AB$ and $BA$ are the same.

   **Hint:** Use Theorem 7 of Lecture 9. Your task is, essentially, to exhibit matrix $S$.

   (b) If $\mathbf{v}$ is an eigenvector of $AB$, what is the corresponding eigenvector of $BA$?

7. **This problem is worth 1.75 regular problems.**

   Let
   
   $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$.

   (a) Find (by hand) the diagonalization of $A$: $A = P \Lambda P^{-1}$, where $\Lambda$ is a diagonal matrix. Exhibit $P$, $P^{-1}$, and $\Lambda$. You may use Mathematica or Matlab to factor the characteristic polynomial.

   (b) Find (with Matlab, see Problem 2) the eigenvectors $\mathbf{u}_j$ of $A^T$. Then verify (either with Matlab or by hand, whichever you prefer) that $\mathbf{u}_j^T \mathbf{v}_i = \delta_{ij}$, where $\mathbf{v}_i$ are the eigenvectors of $A$ and $\delta_{ij}$ is the Kronecker delta.

   State how the $\mathbf{u}_j$’s compare to the rows of $P^{-1}$, where $P$ is defined in part (a).

   (c) By direct calculation, verify the spectral resolution of $A$, i.e. that
   
   $A = \lambda_1 \mathbf{v}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{v}_2 \mathbf{u}_2^T + \lambda_3 \mathbf{v}_3 \mathbf{u}_3^T$.

   Then compute
   
   $B = \mathbf{v}_1 \mathbf{u}_1^T + \mathbf{v}_2 \mathbf{u}_2^T + \mathbf{v}_3 \mathbf{u}_3^T$.  

   $(\dagger)$

   If you use Matlab, you must attach your printout. **(Problem continued on next page.)**
(d) Use the result of part (a) to find \(A^{30}\). (You are allowed to use Matlab or a calculator to multiply large numbers or different matrices, but not to compute \(A^k\).) Write down your answer using 5 significant digits for each entry of the matrix.

State the rank of the matrix that you have written down. (Recall your answer to # 7 of HW 2.)

What is the relation between the columns and rows of your answer for \(A^{30}\) on one hand and the vectors \(v_i\) and \(u_i\) for one of the \(\lambda_i\)’s on the other?

(e) Prove that what you found in part (c) for the matrix \(B\) is not a fluke and will hold for \(v_i\) and \(u_i\) corresponding to any diagonalizable matrix \(A\). To that end, consider an arbitrary vector \(x\) and its expansion over the basis of the eigenvectors of \(A\):

\[
x = c_1v_1 + c_2v_2 + c_3v_3.\]

(††)

Now use the right-hand sides of (†) and (††) to compute \(Bx\) and compare it with \(x\). (Note that you need to use the general relation between vectors \(u_j\) and \(v_i\) that was stated in Lecture 10 and that you verified in part (b) rather than the specific numeric forms of these vectors.) From this comparison between \(Bx\) and \(x\) for an arbitrary \(x\), draw the desired conclusion about any matrix \(B\) having the form of (†).

Note: This important theoretical result is referred to as the spectral resolution of unity.

(f) Finally, solve \(Ax = b\), where \(A\) is given in (a) and \(b = [1, -1, -1]^T\), using the spectral resolution of \(A\) (see the end of Lecture 10).

8. This problem discusses a possible application of the spectral resolution of a matrix in Control theory.

Consider a linear system whose transient behavior is described by

\[
dx/dt = Ax, \tag{HW8.8.1}\]

where \(A\) is a matrix with constant coefficients. We also assume that \(A\) is diagonalizable, i.e. has the spectral resolution given by Eq. (6) of Lecture 10. For simplicity, let us focus on the \(2 \times 2\) case; the general \(p \times p\) case is handled similarly.

Let us assume that one of the eigenvalues of \(A\) is positive, say, \(\lambda_1 > 0\). This implies that this system is unstable: see Eq. (5) and Note 2 after Theorem 6 in Lecture 9. The task of an engineer may be to design a feedback mechanism that would make the system stable. That is, the system should be modified so that its behavior is described by

\[
dx/dt = Ax + f, \tag{HW8.8.2}\]

where the forcing function \(f\) is required to have either of the two forms below:

\[
f_{\text{proportional}} = P \cdot x \tag{HW8.8.3a}\]

or

\[
f_{\text{differential}} = D \cdot \frac{dx}{dt}, \tag{HW8.8.3b}\]

where \(P\) and \(D\) are some matrices. In other words, the engineer’s task is to find the forms of \(P\) and \(D\) that will guarantee that the modified system (HW8.8.2) is stable.

First, note that in the case of proportional forcing (Eq. (HW8.8.3a)), the solution can be achieved by simply taking \(P = cI\), where \(I\) is the identity matrix and \(c\) is a negative number such that \(c < -\lambda_1\). Then by Problem 1(ii) of this set, the eigenvalues of \(A + P = A + cI\) are \((\lambda_1 + c) < 0\) and \((\lambda_2 + c) < 0\) (we assume that \(\lambda_2 < 0\)), which implies that the modified system (HW8.8.2)+(HW8.8.3a) is stable. However, using the differential forcing (HW8.8.3b) with \(D\) being proportional to \(I\) will not make the modified system stable. **Your task** in this problem is to use your knowledge of the spectral resolution of a matrix to propose a form of the differential feedback matrix \(D\) that will make the modified system stable. Follow the steps listed below.
(a) Rewrite system (HW8.8.2)+(HW8.8.3b) as

\[ \frac{dx}{dt} = B A x, \]  

where your task is to relate matrix \( B \) to \( D \).

(b) Suppose you found the eigensystem \( \{ (\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2) \} \) of \( A \) and also the corresponding bi-orthonormal basis \( \{ \mathbf{u}_1, \mathbf{u}_2 \} \). Then

\[ A = \lambda_1 \mathbf{v}_1 \mathbf{u}_1 + \lambda_2 \mathbf{v}_2 \mathbf{u}_2 . \]  

Show that for any \( B \) that has the spectral resolution

\[ B = \mu_1 \mathbf{v}_1 \mathbf{u}_1 + \mu_2 \mathbf{v}_2 \mathbf{u}_2 , \]  

one has

\[ B A = \mu_1 \lambda_1 \mathbf{v}_1 \mathbf{u}_1 + \mu_2 \lambda_2 \mathbf{v}_2 \mathbf{u}_2 . \]

(c) Use your results in parts (a) and (b) to propose a form of matrix \( D \) in (HW8.8.3b) that would stabilize the modified system (HW8.8.2)+(HW8.8.3b).