**Instructions:** Present your work in a neat and organized manner. Please use either the 8.5 × 11 size paper or the filler paper with pre-punched holes. Please do not use paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but not by folding its (upper left) corner.

You must show all of the essential details of your work to get full credit. If I am forced to fill in gaps in your solution by using non-trivial (at my discretion) steps, I will reduce your score for that particular assignment.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting your own work.

All problems in a given assignment contribute equal amount to the assignment’s total score, unless otherwise noted.

**Homework Assignment # 12**

**Due Wednesday, November 19, 2008**

1. **This problem is worth 0.5 of a regular problems.**

   Follow the steps of the derivation of Eq. (23) in Lecture 14 to derive Eq. (25) in that Lecture, assuming no particular initial condition.

   Now, assume that in Eq. (24) $\psi = 0$ and that the initial condition is $x(0) = x_0$, $x'(0) = 0$. Determine the corresponding constants $c_0$ and $\phi_0$ in (25).

2. Work through the steps of Examples 1 and 3 of Lecture 15, as specified below. The goal of this exercise is to remind you of the relation between sine, cosine, and complex exponential functions; see Section 0 in Lecture 1.

   (a) Obtain Eqs. (27a,b) by hand.

   (b) Obtain (29a) from (28a) by hand.

   (c) Obtain (46), as explained in the text. Here you have the choice of either doing the calculations by hand or writing a Mathematica notebook that would compute the vector in (46).

     If you choose to use Mathematica, attach your printout and e-mail me the file.

     **Hint:** If you choose to do calculations by hand, you may find the following identities useful:

     \[
     \cos^2 a = \frac{1 + \cos 2a}{2}, \quad \cos a \cos b + \sin a \sin b = \cos(a - b), \quad \sin a \cos b - \cos a \sin b = \sin(a - b).
     \]

3. Starting with $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, derive Eq. (39) of Lecture 15 using the method of variation of parameters. Follow the steps of this method for a scalar equation, presented in Section 3 of Lecture 14.

   **Note 1:** You do not need to distinguish between the cases where $A$ is diagonalizable or not. All you need to take for granted is that a fundamental solution matrix $\Phi(t)$, which solves

   \[
   \Phi'(t) = A \Phi(t),
   \]

   is available.

   **Note 2:** It is important that you use the correct order of all matrix–matrix and matrix–vector multiplications.

4. (a) Write a Mathematica notebook (or a Matlab file) that computes the solution of Eq. (47) of Lecture 15 for any $\gamma$, $\omega_0$ (where $\gamma < \omega_0$), $f_0$, and $\Omega$. Use Eq. (29a) and Eq. (38) with $\mathbf{x}(0) = \mathbf{0}$.

   The goal of the next two parts is to confirm the validity of the two statements made in the notes after Eq. (47).
(b) Assume $\gamma = 0$, $\omega_0 = 1$, $f_0 = 1$, and $\Omega = 0.95$. Plot $x(t)$ (which is the first component of $\mathbf{x}(t)$) for $0 \leq t \leq 100$ and print out your plot. Circle the portion of the graph that exhibits the resonance-like behavior of the solution.

(c) Assume $\gamma = 0.02$, $\omega_0 = 1$, $f_0 = 1$, and $\Omega = 1$. Plot $x(t)$ for $0 \leq t \leq 100$ and circle the portion of the graph that exhibits the resonance-like behavior of the solution.

5. (a) Demonstrate statement (56) by using only the first three terms in the expansion of $e^X$ (where $X$ is any of $A$, $B$, or $A + B$):

$$e^X = I + X + \frac{1}{2}X^2 + \cdots.$$ 

*Note:* When evaluating the r.h.s. of (56), you will have terms like $AB^2$, $A^2B$, $A^2B^2$, etc. These terms should be omitted since your expansion of the l.h.s. is valid only up to terms like $A^2$, $AB$, etc.

(b) Verify statement (56) by taking (i) $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and (ii) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Use Matlab to compute the matrix exponentials. Note that the r.h.s. should be computed as $\exp(A) \cdot \exp(B)$ rather than as $\exp(A) \times \exp(B)$.

What is the difference between cases (i) and (ii) in light of statement (55)?

6. In the figure shown below, $C_{1,2}$ are the capacitances, $E_{1,2} = E_{10,20} \cos \omega_{1,2} t$ are the voltage sources, $L_{1,2}$ are the inductances, and $M$ is the cross-inductance.

The currents $I_1$ and $I_2$ obey the equations:

$$L_1 I_1'' + M I_2'' + \frac{I_1}{C_1} = E_{10} \cos \omega_1 t$$
$$L_2 I_2'' + M I_1'' + \frac{I_2}{C_2} = E_{20} \cos \omega_2 t,$$

where the prime denotes the time derivative.

(a) First, consider the case $E_1 = E_2 = 0$ for all $t$. Let $\mathbf{I} = [I_1, I_2]^T$. Then (HW12.6.1) can be written as

$$L \mathbf{I}'' + C \mathbf{I} = 0.$$ 

(HW12.6.2)

Seeking a solution of (HW12.6.2) in the form

$$\mathbf{I} = e^{i\omega t} \mathbf{Y},$$ 

(HW12.6.3)

one obtains a generalized eigenvalue problem

$$C \mathbf{Y} = \omega^2 L \mathbf{Y}.$$ 

(HW12.6.4)

From this generalized eigenvalue problem, but *without explicitly computing the eigenvalues*, determine for what relation between $M$ and $L_{1,2}$ the currents will be purely oscillatory.

*Hint 1:* Use a straightforward modification of Problem 5(b) of HW 11. (You do not need to prove that modification; it will suffice to state it.)

*Hint 2:* Under what condition on the coefficient $q$ are both roots of the quadratic equation $x^2 - px + q = 0$ positive, given that $p > 0$?
(b) Now let $E_1 \neq 0$ and $E_2 \neq 0$, so that the currents satisfy
\[ L \frac{d^2 I}{dt^2} + C I = E, \]  
where $E = [E_1, E_2]^T$.
Express $\frac{d^2 I}{dt^2}$ from that equation. Now, using this result, rewrite (HW12.6.5) as
\[ x' = Ax + f, \]  
where $x = [x_1, x_2]^T$, $f = [f_1, f_2]^T$, and $A$ is a matrix.

You need to exhibit the forms of $x$, $f$, and $A$.

*Hint:* See Sections 5 and 6 of Lecture 14.

(c) Under what condition(s) on the voltage sources’ frequencies $\omega_1$ and $\omega_2$ will the currents not be purely oscillatory? (Hint: See p. 17 of Lecture 15.)
- What will they be then (i.e., growing exponentially, linearly, quadratically, etc, or decaying)?
- Under the above condition(s) on $\omega_{1,2}$, do you think that it is possible to have one of the currents oscillatory and the other one not?

(d) This part is optional and is worth 0.25 of a regular problem. Credit will be given only if it is mostly correct. The goals of this part are to show you a new Matlab command and to make you think whether the results you obtain make sense.

Let $L_1 = 1$, $L_2 = 2$, $M = 3$ (Henries), $C_1 = 4$, $C_2 = 5$ (Farads). Use Matlab’s command `polyeig` to find the natural frequencies of the currents from (HW12.6.4). Comment on anything that may have looked strange to you at first sight.

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1I am *not* asking for a calculation, but just for a justified opinion.