Instructions: Present your work in a neat and organized manner. Please use either the 8.5 × 11 size paper or the filler paper with pre-punched holes. Please do not use paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but not by folding its (upper left) corner.

You must show all of the essential details of your work to get full credit. If I am forced to fill in gaps in your solution by using not trivial (at my discretion) steps, I will reduce your score for that particular assignment.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting your own work.

All problems in a given assignment contribute equal amount to the assignment’s total score, unless otherwise noted.

Homework Assignment # 10
Due Friday, November 7, 2008

1. Follow the lines of Example 2 in Lecture 12 to obtain the Jordan canonical form of

\[
A = \begin{pmatrix}
1 & 0 & 3 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}.
\]

Make sure to exhibit matrix \(P\) and the Jordan block \(J\).

No credit will be given for guessing the correct \(x_{10}\); you must find it as it is done in Example 2.

Note that matrix \(A\) is already in the Schur form (see Theorem 3 of Lecture 11), but not in the Jordan canonical form (this is what you need to find).

2. This problem is worth 1.25 regular problems.

(a) Compute \(N^2\) and \(N^3\), where

\[
N = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]

Note: Any matrix \(N\) such that for some \(p > 1\), \(N^{p-1} \neq \Theta\) but \(N^p = \Theta\), where \(\Theta\) is the zero matrix, is called a nilpotent matrix of order \(p\).

(b) Let \(A = PJP^{-1}\), where the \(5 \times 5\) Jordan block \(J\) has the block-diagonal form \(\text{diag}(J_1, J_2, J_3)\), where: \(J_1 = \lambda_1\) (1 × 1 block), \(J_2 = \lambda_2\) (1 × 1 block), and

\[
J_3 = \begin{pmatrix}
\lambda_3 & 1 & 0 \\
0 & \lambda_3 & 1 \\
0 & 0 & \lambda_3
\end{pmatrix}\quad(3 \times 3 \text{ block}).
\]

– What is the characteristic polynomial of \(A\) in the factored form? (Hint: See Theorem 7 and Eq. (9) in Lecture 9.) Let us denote this polynomial \(p_A(\lambda)\).

– Following the lines of Theorem 6 of Lecture 9, derive the factored form of \(p_A(A)\) (i.e., the characteristic polynomial of \(A\) where instead of \(\lambda\), one substitutes \(A = PJP^{-1}\)). Do not yet substitute the explicit form of \(J\).

– Now substitute the explicit form of \(J\) into \(p_A(A)\) of the previous step and show that

\[
p_A(A) = \Theta\quad (\text{the zero matrix}).\quad(*)
\]
Note: Equation (⋆) is known as the Cayley–Hamilton theorem (see p. 361 in the textbook): Any square matrix is a “root” of its own characteristic polynomial.

(c) By direct calculation (you can use Matlab or Mathematica), verify the Cayley–Hamilton theorem for the matrix in Problem 1 above.

3. **This problem is worth 0.5 of a regular problem.**

Compute a few more members of the equation chain (28) in Lecture 12 (say, Eqs. (28.m − 3), (28.m − 4), (28.m − 5)) to verify the pattern of Eq. (28.0) and hence of (29).

4. The time-discrete population model of foxes and chickens considered in Example 4 of Lecture 1 (Example (2.13) in the textbook) can be approximated by a time-continuous differential equation via the following steps.

Let \( \begin{pmatrix} C(t) \\ F(t) \end{pmatrix} \) be the vector of the populations after \( t \) years. Then after the \((t + 1)\)th year, one has:

\[
\begin{pmatrix} C(t+1) \\ F(t+1) \end{pmatrix} = \begin{pmatrix} 1.2 & -k \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} C(t) \\ F(t) \end{pmatrix} \Rightarrow \begin{pmatrix} C(t+1) \\ F(t+1) \end{pmatrix} - \begin{pmatrix} C(t) \\ F(t) \end{pmatrix} = \begin{pmatrix} 0.2 & -k \\ 0.5 & -0.4 \end{pmatrix} \begin{pmatrix} C(t) \\ F(t) \end{pmatrix}.
\]

Dividing both sides of the 2nd equation by \( \Delta t = 1 \) year and remembering that \( (f(t + \Delta t) - f(t))/\Delta t \approx df(t)/dt \), one can approximately rewrite that equation as

\[
\frac{d}{dt} \begin{pmatrix} C(t) \\ F(t) \end{pmatrix} = \begin{pmatrix} 0.2 & -k \\ 0.5 & -0.4 \end{pmatrix} \begin{pmatrix} C(t) \\ F(t) \end{pmatrix}.
\]

(Solve (HW10.4.1) with \( k = 0.18 \) and the initial condition used in the textbook:

\[
x(0) \equiv [C(0), F(0)]^T = [1000, 100]^T.
\]

Follow the approach similar to that which you used in the last problem of HW 9. Namely, do these steps.

(a) Find the Jordan canonical form of \( A: A = PJP^{-1} \). (Matlab can assist you with part of this calculation. The command is still \texttt{eig}, but it will not produce everything that you will need.)

(b) Substitute

\[
x(t) = P \phi(t)
\]

into (HW10.4.1) and obtain the differential equation for \( \phi(t) \).

**Note 1:** Equation (HW.10.4.3a) is equivalent to

\[
x(t) = c_{10}(t)\phi_{10} + c_{11}(t)\phi_{11},
\]

which is similar to the form you used in Problem 6(b) of HW 9.

**Note 2:** Unlike in the aforementioned Problem, here the differential equations for \( c_{10}(t) \) and \( c_{11}(t) \) are not decoupled from each other. **How can you see that from your result?**

**Note 3:** Despite the fact that they are not decoupled, these equations can be easily solved by backward substitution: This is how Eqs. (28) in Lecture 12 were derived (see Problem 3 above). You will need to use the result of that derivation, Eq. (29) of Lecture 12, to complete the solution in part (c) below.

(c) Use Eq. (29) to write down the solution of your equation for \( \phi(t) \) obtained in part (b) with the initial condition given by (HW10.4.2). (The initial condition for \( \phi \) is found from (HW10.4.3a) and (HW10.4.2).) Then use (HW10.4.3a) to obtain \( x(t) \).

(d) Compare your solution to the solution of the original time-discrete model (see the Table in Example (2.17) in the textbook) by plotting both solutions versus the vector of years used in that Table.

Which of the cases depicted on p. 12–19 of the lecture notes does this solution belong to?
5. This problem is worth 1.25 regular problems.

Introduction

The main goal of this Problem is to present a simple mechanical analogy of the generalized eigenvector. The secondary goal is to practice the approach of Problem 6 of HW 9 and Problem 4 of this set, where one seeks the solution of a system of coupled equations in the form of a superposition of decoupled (as in Problem 6 of HW 9) or coupled-in-a-simpler-fashion (as in Problem 4 above) modes.

To illustrate the first goal, consider the simplest possible equation of motion for a single particle:

\[ x''(t) = 0, \quad x(0) = x_0, \quad x'(0) = y_0. \quad \text{(HW10.5.1)} \]

\((y_0\) is, of course, the initial velocity, but the notation \(v\) is already used for eigenvectors; therefore, in this Problem we will denote all velocity-type variables by \(y\).) Equation (HW10.5.1) can be rewritten as a linear system

\[ \mathbf{x} \equiv \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{x}'(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \text{(HW10.5.2)} \]

where the matrix — let us call it \(J\) — is clearly defective and is already in the Jordan canonical form. To emphasize the analogy with the approach used in Problem 4 (see the secondary goal stated above), let us note that

\[ \mathbf{x}(t) = x(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv c_{10}(t)\mathbf{y}_1 + c_{11}(t)\mathbf{y}_1 + (t). \quad \text{(HW10.5.3)} \]

That is, the position \(x(t)\) of the particle is the coordinate of the eigenvector, and the velocity \(y(t)\) is the coordinate of the generalized eigenvector. We will, therefore, use the notations \(c_{10}(t) = x(t)\) and \(c_{11}(t) = y(t)\) interchangeably to emphasize this analogy.

To complete the solution of (HW10.5.2), we use Eq. (29) of Lecture 12:

\[ \begin{pmatrix} c_{10}(t) \\ c_{11}(t) \end{pmatrix} = e^{tJ} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} e^{0t} & t e^{0t} \\ 0 & e^{0t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \Rightarrow \quad \begin{cases} x(t) = x_0 + y_0t, \\ y(t) = y_0, \end{cases} \quad \text{(HW10.5.4)} \]

as expected. Thus, the fact that \{ the solution of equation \(\dot{\mathbf{x}} = J\mathbf{x}\) with a defective matrix \(J\) grows linearly in time along the direction of the eigenvector \} is analogous to the well-known fact that \{ the coordinate of a particle moving with a constant speed grows linearly in time along the direction of motion \}.

Assignment

Consider two identical masses \(m\) connected by a spring with the constant \(k\) (this is a simpler version of the setup of Problem 6 of HW 9). Neither mass is connected to a wall. Denote \(k/m = \omega^2\).

(a) Derive the equations of motion for the positions \(X_1\) and \(X_2\) of the masses and then verify that vector \(\mathbf{x} = [X_1, Y_1, X_2, Y_2]^T\), where \(Y_j = X_j'\), satisfies the matrix equation

\[ \mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 & 0 & -\omega^2 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} X_1(0) \\ X_1'(0) \\ X_2(0) \\ X_2'(0) \end{pmatrix}. \quad \text{(HW10.5.5)} \]

Denote the matrix in (HW10.5.5) by \(A\).
(b) The Jordan canonical form of $A$ is: $A = P J P^{-1}$, where

$$ P = \begin{pmatrix} 1 & 0 & \frac{i}{\sqrt{2}\omega} & \frac{i}{\sqrt{2}\omega} \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -\frac{i}{\sqrt{2}\omega} & -\frac{i}{\sqrt{2}\omega} \\ 0 & 1 & -1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{2}\omega & 0 \\ 0 & 0 & 0 & i\sqrt{2}\omega \end{pmatrix}. \quad (HW10.5.6) $$

Follow the method given in the above Introduction (which is analogous to the method in Problem 6 of HW 9) and in Example 4 of Lecture 12 to obtain the solution of the ODE in (HW10.5.5).

*Note* that the equations for three out of four coordinates $c_{10}(t)$, $c_{11}(t)$, $c_{2}(t)$, and $c_{3}(t)$ are decoupled, and the remaining equation is coupled to the rest of them in a very simple way.

(c) Find the initial condition for the coordinates $c_{10}$, $c_{11}$, $c_{2}$, and $c_{3}$ using the initial condition for vector $\mathbf{x}$ given in (HW10.5.5).

Finally, obtain the solution $\mathbf{x}(t)$ of the initial-value problem (HW10.5.5).

*Hint:* See Example 4 in Lecture 12.

(d) Looking at the form of this solution, interpret it as a superposition of a few modes of motion. How many of these modes there are and what are they? (See the end of Problem 6 in HW 9 for a similar discussion.)