11. Tower of Hanoi

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The Tower of Hanoi: Origin

In the great temple at Benares, beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four discs of pure gold, the largest disk resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Bramah. Day and night unceasingly the priests transfer the discs from one diamond needle to another according to the fixed and immutable laws of Bramah, which require that the priest on duty must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When the sixty-four discs shall have been transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple, and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish.

The Tower of Hanoi: The Puzzle

Initial State:

Goal State:

Only one disk can be moved at a time, from one peg to another, such that it never is placed on a disk of smaller diameter.
1 Move.
Two Disk Solution

Note that we perform the solution to the one-disk Tower of Hanoi twice (once in the top row, and once in the bottom row). Between rows, we move the bottom disk. Thus, we require

$$2 \cdot 1 + 1 = 3 \text{ Moves}$$
Note that we perform the solution to the two-disk Tower of Hanoi *twice* (once in the top row, and once in the bottom row). Between rows we move the bottom disk. Thus, we require

\[ 2(2 \cdot 1 + 1) + 1 = 7 \text{ Moves} \]
Tower of Hanoi: What we know so far

Let $M(n)$ denote the minimum number of legal moves required to complete a tower of Hanoi puzzle that has $n$ disks.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

Following the pattern, for $n = 4$ we need to solve the three-disk puzzle twice, plus one more operation to move the largest disk. Thus,

$$M(4) = 2 \cdot M(3) + 1 = 2 \cdot 7 + 1 = 15.$$  

Similarly, for $n = 5$ disks, we expect that we will need to perform

$$M(5) = 2 \cdot M(4) + 1 = 2 \cdot 15 + 1 = 31.$$
Tower of Hanoi: \( n \) Disk Analysis

Let \( M(n) \) denote the minimum number of legal moves required to complete a tower of Hanoi puzzle that has \( n \) disks.

- Before the largest disk (i.e., the \( n \)-th disk) can be moved to the rightmost peg, all of the remaining \((n - 1)\) disks must moved to the center peg. (These \( n - 1 \) disks must be somewhere, and they can’t obstruct the transfer of the largest disk.) This requires \( M(n - 1) \) legal moves.
- It takes 1 more operation to move the \( n \)-th disk to the rightmost peg.
- Finally, another legal sequence of \( M(n - 1) \) steps is required to move the \( n - 1 \) disks from the center peg, to the rightmost peg.

We thus obtain the recursion relation,

\[
M(n) = 2M(n - 1) + 1.
\]
Tower of Hanoi: Solution

With the solution for a single disk

\[ M(1) = 1 \]

the recursion relation

\[ M(n) = 2M(n - 1) + 1. \]

defines the solution

\[ M(n) = 2^n - 1. \]

In a hierarchy of algorithms, this would be called \textit{exponential} or \( O(2^n) \).
The practical difficulty with exponential algorithms is that they can quickly grow out of hand. (With each additional disk, the minimum number of operations essentially doubles.) N.B., 1 century \( \approx 4.5 \times 10^9 \) seconds. (The age of the universe is \( 4 \times 10^{17} \) seconds.)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2^n - 1 )</th>
<th>( n )</th>
<th>( 2^n - 1 )</th>
<th>( n )</th>
<th>( 2^n - 1 )</th>
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</tr>
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</table>
Tower of Hanoi: Strategy

We have seen that the solution to this puzzle is recursive:

- In order to move all $n$ disks to the right peg, we must first move the top $n - 1$ disks to the center peg.
- Before this, we must move the top $n - 2$ disks to the right peg.
- And before this, we must move the top $n - 3$ disks to the center peg, and so on.

So what should the first move be? Should the top disk be moved to the center peg, or to the right peg?
If the number of disks is *odd*, then the first move should be to transfer the top disk to the *right* peg.

If the number of disks is *even*, then the first move should be to transfer the top disk to the *center* peg.

The sequence of states that is visited in the course of solving the puzzle is called the *solution path*.

The length of the shortest solution path to the *n*-disk puzzle is $2^n$. 
A configuration of disks in the Tower of Hanoi puzzle is said to be a legal state if no disk rests on a disk of smaller diameter. That is, the largest disk on each peg must be placed on the bottom, and the remaining disks must be placed in the order of decreasing diameter.

Question: How many legal states are there for \( n \) disks placed on three pegs?
Tower of Hanoi: Accessible States

A configuration of disks is said to be an **accessible state**, if it can be realized from the initial state after a legal sequence of moves.

Question: How many accessible states are there for \( n \) disks placed on **three** pegs?
## Tower of Hanoi: Examples

<table>
<thead>
<tr>
<th>Number of Disks $n$</th>
<th>Length of Solution Path $2^n$</th>
<th>Number of Legal States $3^n$</th>
<th>Fraction of Visited States $(2/3)^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0.6666667</td>
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<tr>
<td>2</td>
<td>4</td>
<td>9</td>
<td>0.4444446</td>
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<tr>
<td>3</td>
<td>8</td>
<td>27</td>
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<tr>
<td>4</td>
<td>16</td>
<td>81</td>
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<td>32</td>
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<td>1853020188851841</td>
<td>$2.32 \times 10^{-6}$</td>
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</tbody>
</table>
State Representation

It’s cumbersome to represent a state by an illustration. Instead we will adopt a list notation.
For example, for a five disk puzzle, the notation

\[(2 \ 1 \ 3 \ 1 \ 1)\]

indicates

- The smallest disk is on peg 2 (the center peg).
- The next larger disk is on peg 1 (the left peg).
- The next larger disk is on peg 3 (the right peg).
- The next larger disk is on peg 1.
- The next larger (i.e., largest) disk is on peg 1.

The only legal configuration that describes this is
State Graph: 1 disk

Each state is represented by a labeled *vertex*; legal moves are represented by *edges*. 
State Graph: 2 disks

(11)

(21) — (31)

(23) — (32)

(33) — (13) — (12) — (22)
State Graph: 3 disks