

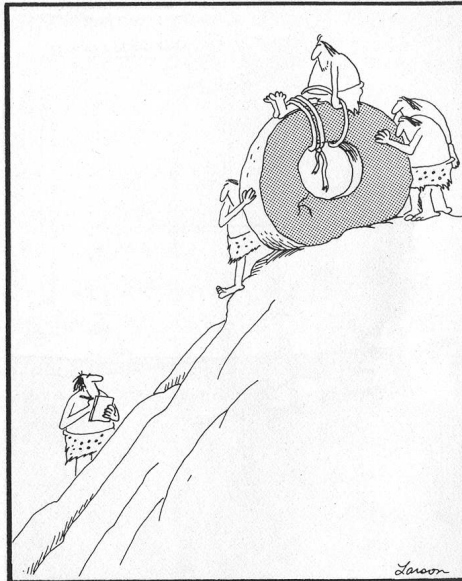
Electrostatic Energy

EE 141 Lecture Notes
Topic 14

Professor K. E. Oughstun
School of Engineering
College of Engineering & Mathematical Sciences
University of Vermont

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Motivation



Early experiments in transportation

Potential Energy of a Static Charge Distribution

Whenever two charges q_a and q_b are brought within a distance R_{ab} of each other, work is expended against the Coulomb force [see Topic 1, Eq. (3)] in consummating the process. Once the charges are in place, the persistence of the Coulomb force makes the energy stored in the electrostatic field potentially available whenever demanded.

If it is assumed that the charges are moved slowly enough into place (i.e. reversibly), then their kinetic energies may be neglected and any loss due to electromagnetic radiation effects, significant if rapid charge accelerations occur, may be neglected.

Potential Energy of a Static Charge Distribution

Consider the energy stored in a fixed configuration of n charges, given by the work required to assemble the charge configuration.

- Assume that all n charges q_1, q_2, \dots, q_n are initially located at infinity in their zero potential state.
- Upon bringing just q_1 from infinity to its final position P_1 , no work is expended because no other charges are present.
- The work done in bringing q_2 from infinity to P_2 is given by

$$U_2 = q_2 V_2^{(1)} = q_2 \frac{q_1}{4\pi\epsilon_0 R_{12}} \quad (1)$$

$$= q_1 V_1^{(2)} = q_1 \frac{q_2}{4\pi\epsilon_0 R_{21}} \quad (2)$$

where $V_2^{(1)}$ denotes the electrostatic potential at P_2 due to the charge q_1 at P_1 , and where $V_1^{(2)}$ denotes the electrostatic potential at P_1 due to the charge q_2 at P_2 .

Potential Energy of a Static Charge Distribution

- The work done in bringing a third charge q_3 in from infinity to P_3 is then given by

$$U_3 = q_3 V_3^{(1)} + q_3 V_3^{(2)} \quad (3)$$

$$= q_1 V_1^{(3)} + q_2 V_2^{(3)}, \quad (4)$$

and so on for the remaining charges q_4, q_5, \dots, q_n , noting that

$$\boxed{q_k V_k^{(j)} = q_j V_j^{(k)}} \quad (5)$$

where $V_k^{(j)}$ denotes the electrostatic potential at P_k due to the charge q_j at P_j .

Potential Energy of a Static Charge Distribution

- The total energy $U_e = U_1 + U_2 + \dots + U_n$ can be written in two different ways: First by adding Eqs. (1), (3), etc., giving

$$\begin{aligned}U_e = & q_2 V_2^{(1)} \\ & + q_3 V_3^{(1)} + q_3 V_3^{(2)} \\ & + q_4 V_4^{(1)} + q_4 V_4^{(2)} + q_4 V_4^{(3)} \\ & + \dots + q_n V_n^{(1)} + q_n V_n^{(2)} + \dots + q_n V_n^{(n-1)}, \quad (6)\end{aligned}$$

- or by adding Eqs. (2), (4), etc., giving

$$\begin{aligned}U_e = & q_1 V_1^{(2)} \\ & + q_1 V_1^{(3)} + q_2 V_2^{(3)} \\ & + q_1 V_1^{(4)} + q_2 V_2^{(4)} + q_3 V_3^{(4)} \\ & + \dots + q_1 V_1^{(n)} + q_2 V_2^{(n)} + \dots + q_{n-1} V_{n-1}^{(n)}. \quad (7)\end{aligned}$$

Potential Energy of a Static Charge Distribution

Adding Eqs. (6) and (7) together and dividing by 2 then yields the more symmetric expression

$$U_e = \frac{1}{2} \left\{ q_1 \left[V_1^{(2)} + V_1^{(3)} + V_1^{(4)} + \cdots + V_1^{(n)} \right] \right. \\ + q_2 \left[V_2^{(1)} + V_2^{(3)} + V_2^{(4)} + \cdots + V_2^{(n)} \right] \\ + q_3 \left[V_3^{(1)} + V_3^{(2)} + V_3^{(4)} + \cdots + V_3^{(n)} \right] \\ \left. + \cdots + q_n \left[V_n^{(1)} + V_n^{(2)} + V_n^{(3)} + \cdots + V_n^{(n-1)} \right] \right\}.$$

Potential Energy of a Static Charge Distribution

The potential energy of the assembled charge configuration is then given by

$$U_e = \frac{1}{2} \sum_{k=1}^n q_k V_k \quad (8)$$

where

$q_k =$ charge of the k^{th} particle located at P_k ,

$V_k =$ absolute potential at P_k due to all of the charges except q_k .

Notice that this expression does not include the self-energy of the individual charges; this is the energy that would be liberated if each charge was allowed to expand to an infinite volume. As a consequence, Eq. (8) identically vanishes for a single point charge.

Potential Energy of a Static Charge Distribution

For a continuous (macroscopic) volume charge distribution $\rho_v(\mathbf{r})$, the expression (8) for the electrostatic potential energy generalizes to

$$U_e = \frac{1}{2} \iiint_V \rho_v(\mathbf{r}) V(\mathbf{r}) d^3r \quad (9)$$

For a continuous (macroscopic) surface charge distribution $\rho_s(\mathbf{r})$, the expression (8) for the electrostatic potential energy generalizes to

$$U_e = \frac{1}{2} \iint_S \rho_s(\mathbf{r}) V(\mathbf{r}) d^2r \quad (10)$$

For a continuous (macroscopic) line charge distribution $\rho_\ell(\mathbf{r})$, the expression (8) for the electrostatic potential energy generalizes to

$$U_e = \frac{1}{2} \int_C \rho_\ell(\mathbf{r}) V(\mathbf{r}) d\ell \quad (11)$$

Notice that these expressions include the self-energies of the charges.

Self-Energy of a Spherical Charge Distribution

For a uniform spherical charge distribution of charge Q and radius r_0 with charge density $\rho_v = 3Q/(4\pi r_0^3)$ for $r \leq r_0$, the absolute electrostatic potential inside the sphere is given by (see Topic 7)

$$V(r) = \frac{Q}{8\pi\epsilon_0 r_0^3} (r_0^2 - r^2) + \frac{Q}{4\pi\epsilon_0 r_0}; \quad r \leq r_0.$$

From Eq. (9), the self-energy of this spherical charge distribution is

$$\begin{aligned} U_{se} &= \frac{1}{2} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^{r_0} \rho_v(r) V(r) r^2 dr \\ &= \frac{3Q}{2r_0^3} \int_0^{r_0} \left[\frac{Q}{8\pi\epsilon_0 r_0^3} (r_0^2 - r^2) + \frac{Q}{4\pi\epsilon_0 r_0} \right] r^2 dr \\ &= \frac{3Q^2}{20\pi\epsilon_0 r_0} \rightarrow \infty \text{ as } r_0 \rightarrow 0 \text{ at fixed } Q > 0. \end{aligned}$$

On the other hand,

$$U_{se} = \frac{\rho_v}{\epsilon_0} r_0^2 \rightarrow 0 \text{ as } r_0 \rightarrow 0 \text{ at fixed } \rho_v.$$

Electrostatic Energy

From [Poisson's equation](#)

$$\rho_v(\mathbf{r}) = -\epsilon \nabla^2 V(\mathbf{r}) \quad (12)$$

at every point in the electrostatic field in a material with dielectric permittivity ϵ . Substitution of this expression in Eq. (9) then gives

$$U_e = -\frac{\epsilon}{2} \iiint_{\mathcal{V}} V(\mathbf{r}) \nabla^2 V(\mathbf{r}) d^3r, \quad (13)$$

where \mathcal{V} is any volume containing all of the charges in the system. From [Green's first integral identity](#) [see Eq. (40) of Topic 2]

$$\iiint_{\mathcal{V}} (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d^3r = \oint_S \phi \nabla \psi \cdot \hat{\mathbf{n}} d^2r$$

with $\phi(\mathbf{r}) = \psi(\mathbf{r}) = V(\mathbf{r})$, one obtains

$$\iiint_{\mathcal{V}} (V \nabla^2 V + (\nabla V)^2) d^3r = \oint_S V \nabla V \cdot \hat{\mathbf{n}} d^2r$$

Electrostatic Energy

With this substitution, the expression (13) for the electrostatic energy becomes

$$U_e = -\frac{\epsilon}{2} \left[\oint_S V \nabla V \cdot \hat{\mathbf{n}} d^2r - \iiint_{\mathcal{V}} (\nabla V)^2 d^3r \right] \quad (14)$$

Note that:

- 1 because \mathcal{V} can be any volume that contains all of the charges in the system configuration, the boundary surface \mathcal{S} may then be chosen at an arbitrarily large distance from the charge distribution;
- 2 because $V(\mathbf{r})$ falls off at least as fast as $1/r$ as $r \rightarrow \infty$, then $\nabla V(\mathbf{r})$ falls off at least as fast as $1/r^2$ as $r \rightarrow \infty$, and because the surface area of \mathcal{S} increases as r^2 in that limit, then the surface integral appearing in Eq. (14) decreases at least as fast as $1/r$ as $r \rightarrow \infty$ and can be made arbitrarily small by choosing \mathcal{S} sufficiently distant from the source charge distribution.

Electrostatic Energy

Because $\nabla V(\mathbf{r}) = -\mathbf{E}(\mathbf{r})$, the **electrostatic energy** is then given by

$$U_e = \frac{\epsilon}{2} \iiint_{\mathcal{V}} E^2(\mathbf{r}) d^3r = \frac{1}{2} \iiint_{\mathcal{V}} \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) d^3r \quad (15)$$

where the volume \mathcal{V} must now only be large enough to include all regions where the electrostatic field $\mathbf{E}(\mathbf{r})$ produced by the charge distribution is nonzero. If this is not satisfied, then the electrostatic energy is given by Eq. (13). **Notice that this expression includes the self-energies of all the charges in the system and is positive-definite, whereas the expression given in Eq. (8) can be negative.**

The integrand appearing in Eq. (15) is defined as the **electrostatic energy density**

$$u_e(\mathbf{r}) \equiv \frac{1}{2} \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \quad (J/m^3) \quad (16)$$

which is associated with the field energy at each point in the field.