EE 171

(October 23, 2011)

Signals & Systems

Fall Semester 2011

HOMEWORK No. 6 & 7
Due Thu. Oct. 27, 2011

1. Problem 2.17, Solution of first-order differential equation.
2. Problem 2.18, Impulse response of first-order difference equation.
3. Problem 2.32, Solution of first-order difference equation.
4. Problem 2.61 (a), (b), (c). Do part (i) for 2.61(a), (b) and parts (i), (ii) for 2.61(c).
5. Problem 3.1 Periodic signal from Fourier series coefficients.
6. Problem 3.3 Fourier series of sum of 2 signals.
7. Problem 3.4 Fourier series of periodic square wave.
8. Problem 3.5 Fourier series properties.
9. Problem 3.7 Fourier series properties.

EE-171 HW 6 & 7 SOLUTIONS

[redacted] 2.17(a),(b) 2.18, 2.32, 2.61 (a),(b),(c) [redacted]

3.1, 3.3, 3.4, 3.5, 3.7

2.17(a)

$\because y(t) = \underbrace{y_p(t)}_{\text{particular solution}} + \underbrace{y_h(t)}_{\text{homogeneous solution}}$

$\therefore$ ① Find $y_p(t)$:

Hypothesize: $y_p(t) = K e^{(-1+3j)t}$, $t > 0$

Substitute $x(t)$ and $y(t)$ into the given differential equation:

$$(-1+3j) K e^{(-1+3j)t} + 4K e^{(-1+3j)t} = e^{(-1+3j)t}$$

$$(-1+3j)K + 4K = 1 \qquad \therefore K = \frac{1}{3+3j}$$

$$\boxed{\therefore y_p = \frac{1}{3+3j} e^{(-1+3j)t}}$$

② Find $y_h(t)$:

Hypothesize: $y_h(t) = A e^{st}$.

The homogenous ~~solution~~ should satisfy $\frac{d y_h(t)}{dt} + 4 y_h(t) = 0$.

$$\therefore s A e^{st} + 4 A e^{st} = 0 \qquad \therefore s = -4$$

$$\therefore \boxed{y_h(t) = A e^{-4t}} \qquad \therefore \boxed{y(t) = y_p(t) + y_h(t) = \frac{1}{3+3j} e^{(-1+3j)t} + A e^{-4t}}$$

$\because$ System is rest at initial $\therefore y(0) = 0$

$$\therefore \frac{1}{3+3j} + A = 0 \qquad \boxed{A = -\frac{1}{3+3j}}$$

$$\therefore y(t) = \frac{1}{3+3j} e^{(-1+3j)t} - \frac{1}{3+3j} e^{-4t}$$

$$= \frac{1}{3+3j} \left[e^{(-1+3j)t} - e^{-4t} \right]$$

$\because$ Initial rest. $\therefore y(t) = 0$, when $t \le 0$

$$\boxed{\therefore y(t) = \frac{1}{3+3j} \left[e^{(-1+3j)t} - e^{-4t} \right] \cdot u(t)}$$

b)

$$\frac{d}{dt}y(t) + 4y(t) = x(t)$$

$$\therefore Re\left\{\frac{dy(t)}{dt} + 4y(t)\right\} = Re\{x(t)\}$$

$$Re\left\{\frac{dy(t)}{dt}\right\} + 4\,Re\{y(t)\} = Re\{x(t)\}$$

Suppose $y(t) = r(t) + j\,i(t)$, $r(t)$ and $i(t)$ are real functions.

$$\therefore \frac{dy(t)}{dt} = \frac{d[r(t)+j\cdot i(t)]}{dt} = \frac{dr(t)}{dt} + j\frac{di(t)}{dt}$$

$\because \frac{dr(t)}{dt}$, $\frac{di(t)}{dt}$ are real, $r(t) = Re\{y(t)\}$

$$\therefore Re\left\{\frac{dy(t)}{dt}\right\} = \frac{dr(t)}{dt} = \frac{d}{dt}\left[Re\{y(t)\}\right]$$

$$\therefore \frac{d}{dt}\left[Re\{y(t)\}\right] + 4\cdot Re\{y(t)\} = Re\{x(t)\}$$

When $x(t)$ changes to be $Re\{x(t)\}$, $y(t)$ will change to $Re\{y(t)\}$.

$$\therefore \text{Now } y(t) = Re\{y_{old}(t)\} = Re\left\{\frac{1}{3+3j}\left[e^{(-1+3j)t} - e^{-4t}\right]u(t)\right\}$$

$$= Re\left\{-\frac{1-j}{6}e^{-4t} + \frac{1-j}{6}e^{-t}\cdot e^{3tj}\right\}u(t)$$

$$= Re\left\{-\frac{1}{6}e^{-4t} + j\frac{1}{6}e^{-4t} + \frac{1}{6}e^{-t}(\cos 3t + j\sin 3t) - \frac{j}{6}e^{-t}(\cos 3t + j\sin 3t)\right\}u(t)$$

$$= \left[-\frac{1}{6}e^{-4t} + \frac{1}{6}e^{-t}\cos 3t + \frac{1}{6}e^{-t}\sin 3t\right]u(t)$$

$$= \frac{1}{6}\left[e^{-t}\cos 3t + e^{-t}\sin 3t - e^{-4t}\right]u(t)$$

2.18. Since $x[n] = \delta[n-1] = \begin{cases} 1, & n=1 \\ 0, & \text{elsewhere} \end{cases}$

$\therefore y[n] = 0$, $n \le 0$

$y[1] = 1$

$y[2] = \frac{1}{4}$

$y[3] = (\frac{1}{4})^2$

$\vdots$

$$\therefore y[n] = \left(\frac{1}{4}\right)^{n-1}u[n-1].$$

2-18 $\quad y[n] = \frac{1}{4} y[n-1] + x[n]$

$$x[n] = \delta[n-1] = \begin{cases} 1, & n=1 \\ 0, & \text{elsewhere} \end{cases}$$

System causal.

Method 1.

$\therefore \; y[n] = 0 \;, \; n \le 0$

$y[1] = 1$

$y[2] = \frac{1}{4}$

$y(3) = \left(\frac{1}{4}\right)^2 \quad \cdots \quad \therefore \boxed{y[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]}$

Method 2

$y[n] - \frac{1}{4} y[n-1] = \delta[n-1]$

for $n > 1 \quad y[n] - \frac{1}{4} y[n-1] = 0$

Try $y[n] = K\alpha^n$

$\therefore \; K\alpha^n - \frac{1}{4} K\alpha^{n-1} = 0 \quad \therefore K\alpha^n(1 - \frac{1}{4}\alpha^{-1}) = 0$

Yes if $\alpha = \frac{1}{4}$

$\therefore \; y[n] \equiv y_H(n) = K\left(\frac{1}{4}\right)^n \quad n > 1 \quad$ — (1)

Now $y(1) = \frac{1}{4} y(-1) + 1$

Assuming $y(-1) = 0$ get $y(1) = 1$

$y[2] = \frac{1}{4} y(1) = \frac{1}{4}$

$\therefore$ from (1) $\quad y(2) = \frac{1}{4} = K\left(\frac{1}{4}\right)^2 \quad \therefore K = \left(\frac{1}{4}\right)^{-1}$

$\therefore \; y[n] = \left(\frac{1}{4}\right)^{-1}\left(\frac{1}{4}\right)^n \quad n \ge 1 \quad \boxed{= \left(\frac{1}{4}\right)^{n-1} u[n-1]}$

2.32 $\quad y[n] - \frac{1}{2} y[n-1] = x[n] = \left(\frac{1}{3}\right)^n u[n]$

$$y[n] = y_H[n] + y_P[n]$$

(a) $\underline{y_H[n]}$ Try $y_H[n] = K\alpha^n$ as solution to homogeneous equation $y[n] - \frac{1}{2} y[n-1] = 0$

So: $K\alpha^n - \frac{1}{2} K\alpha^{n-1} = K\alpha^n\left(1 - \frac{1}{2}\alpha^{-1}\right) \overset{?}{=} 0$ Yes, if $\alpha = \frac{1}{2}$

$$\therefore \quad y_H[n] = K\left(\frac{1}{2}\right)^n$$

(b) $y_P[n]$. non-homogeneous equation

$$y[n] - \frac{1}{2} y[n-1] = \left(\frac{1}{3}\right)^n u[n]$$

Try $y[n] = A\left(\frac{1}{3}\right)^n u[n]$.

So, for $n \geq 0$

$$A\left(\frac{1}{3}\right)^n - \frac{A}{2}\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^n$$

$\therefore \quad \cancel{A\left(\frac{1}{3}\right)^n}\left[1 - \left(\frac{1}{3}\right)^{-1}\frac{1}{2}\right] = \cancel{\left(\frac{1}{3}\right)^n} \qquad A\left(\frac{-1}{2}\right) = 1 \quad \underline{A = -2}$

$$\therefore \quad y_P[n] = -2\left(\frac{1}{3}\right)^n u[n]$$

(c) Total solution

$$y[n] = K\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n \qquad n \geq 0$$

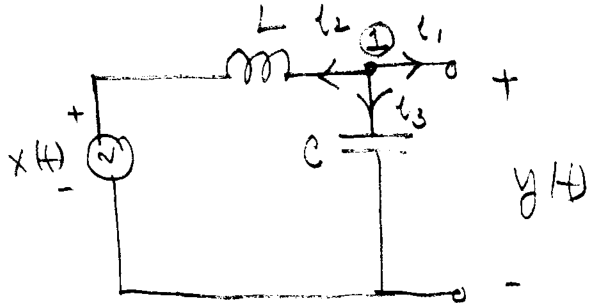
$$\therefore \boxed{y[n] = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n \quad n \geq 0}$$

$$y[0] - \frac{1}{2} y[-1] = 1$$

$\therefore \quad \underline{y[0] = 1 = K - 2} \quad \therefore \quad K = 3$

2.61

a)



(i) KCL node ① $\quad i_1 + i_2 + i_3 = 0$

$$\therefore C\frac{dy}{dt} + \frac{1}{L}\int_{-\infty}^{t} (y(t) - x(t))\,dt = 0$$

$$\boxed{C\frac{d^2y}{dt^2} + \frac{1}{L}y(t) = \frac{1}{L}x(t)}$$

(ii) $C\dfrac{d^2y}{dt^2} + \dfrac{1}{L}y(t) = 0$ or $\dfrac{d^2y}{dt^2} + \dfrac{1}{LC}y = 0$ homogeneous equation

Assume homogeneous solution $y(t) = Ke^{st}$

$\therefore Ke^{st}\left(s^2 + \frac{1}{LC}\right) = 0 \Rightarrow \boxed{s^2 + \frac{1}{LC} = 0}$ characteristic equation

$\Rightarrow s_{1,2} = \pm \dfrac{j}{\sqrt{LC}}$

$\therefore$ solution $y(t) = K_1 e^{+j\frac{1}{\sqrt{LC}}t} + K_2 e^{-j\frac{1}{\sqrt{LC}}t}$

$\omega_1 = \frac{1}{\sqrt{LC}} \quad \omega_2 = -\frac{1}{\sqrt{LC}}$

(iii) Since $y(t)$ real, must have $K_2 = K_1^*$

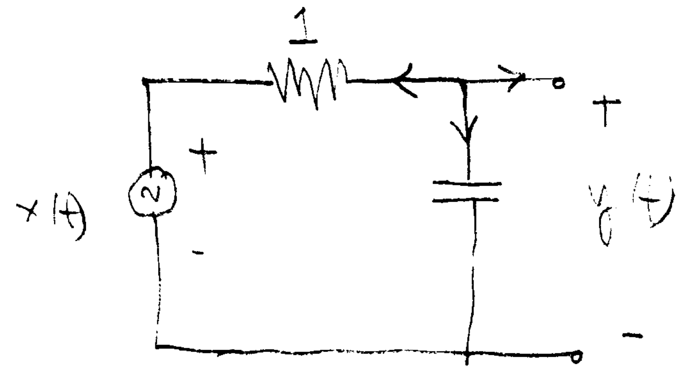
$$\therefore y(t) = K_1 e^{j\frac{1}{\sqrt{LC}}t} + \left(K_1 e^{j\frac{1}{\sqrt{LC}}t}\right)^*$$

$$= 2\,\mathrm{Re}\left\{K_1 e^{j\frac{1}{\sqrt{LC}}t}\right\}$$

Let $K_1 = |K_1|e^{j\theta_1}$

$$\boxed{\therefore y(t) = 2|K_1|\cos\left(\frac{1}{\sqrt{LC}}t + \theta_1\right)}$$

(b)



(i) $C\frac{dy}{dt} + \frac{y(t) - x(t)}{R} = 0 \quad \Rightarrow \quad \boxed{C\frac{dy}{dt} + \frac{y(t)}{R} = \frac{x(t)}{R}}$

(ii) Homogeneous solution

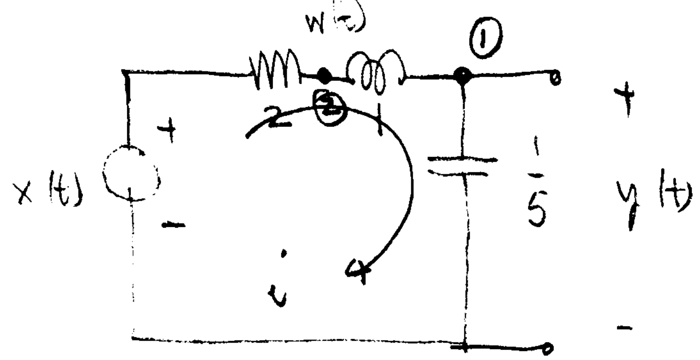
$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = 0 \quad . \quad \text{Assume solution } Ke^{st}$$

char. eqn $\quad s + \frac{1}{RC} = 0 \quad$ natural frequency $s = -\frac{1}{RC} = -1$

$\therefore$ homogeneous solution is natural response

$$y(t) = Ke^{-\frac{1}{RC}t} = Ke^{-1t} \qquad \boxed{a = -1}$$

(c)



(i) $x(t) = 2i(t) + 1\frac{di(t)}{dt} + y(t) \qquad (1)$

$y(t) = 5\int_{-\infty}^{t} i(t)\,dt$

$\therefore \frac{1}{5}\frac{dy}{dt} = i(t) \qquad -(2)$

$\therefore$ from (1),

$$x(t) = \frac{2}{5}\frac{dy}{dt} + \frac{1}{5}\frac{d^2y}{dt^2} + y(t)$$

$$\boxed{\therefore \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y(t) = 5x(t)}$$

(ii) $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$$

Char. eqn: $s^2+2s+5=0$ $\quad s_{1,2} = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm j2$

$$\therefore y_{Trans.}(t) = K_1 e^{(-1+j2)t} + K_2 e^{(-1-j2)t}$$

$$= e^{-t}\left(K_1 e^{j2t} + K_2 e^{-j2t}\right) \qquad \boxed{\alpha = -1}$$

(iii) Since $y_{Trans}(t)$ real, must have $K_2 = K_1^*$

$$\therefore y_{Trans}(t) = e^{-t}\,\mathrm{Re}\{K_1 e^{j2t}\}$$

Let $K_1 = |K_1| e^{j\theta}$

$$\boxed{\therefore y_{Trans}(t) = |K_1| e^{-t}\cos(2t+\theta)}$$

a decaying sinusoid.