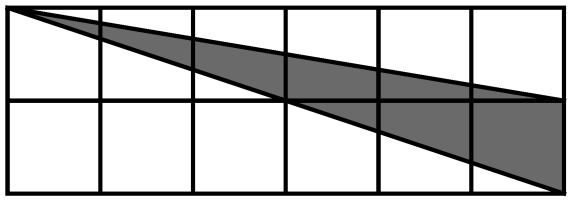
UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
FIFTY-SECOND ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 11, 2009

1) Express $\dfrac{\left(\frac{7}{8}-\frac{1}{16}\right)^2}{\left(\frac{7}{8}\right)^2-\left(\frac{1}{16}\right)^2}$ as a rational number in lowest terms.

2) If $\sqrt{x+2}=3$, what is the value of $(x+3)^2$?

3) A rectangle consists of 12 squares, each of which has an edge length of 1 inch. Find the area of the shaded region.



4) The hour hand of a certain clock is 5 inches long. The minute hand of the clock is 7 inches long. What is the distance between the tips of the hands of the clock at 4 p.m.?

5) Find the value of $\sin^2(15°)\cos^2(15°)$. Express your answer as a rational number in lowest terms.

6) Express $4.\overline{81}$ as a rational number in lowest terms.

7) Krisan can rake her yard in seven hours. Patti can rake the same yard in half that time while Nick would take four times as long as Patti to rake the yard. If they all work together, how long will it take them to rake Krisan's yard?

8) Let x, y and z be positive integers satisfying $x^2+y^2+z^2=6$. Find $x+y+z$.

9) When the integer 7^{2009} is written as a binary number, what are the three rightmost digits of the binary number?

10) The Elk River Band is selling cheesecakes to raise money for a band trip to New Orleans. A large cheesecake sells for \$10.50 and a small cheesecake sells for \$8.00. If the total sales were \$945 and the ratio of large cheesecakes to small cheesecakes was 2 to 3, how many large cheesecakes were sold?

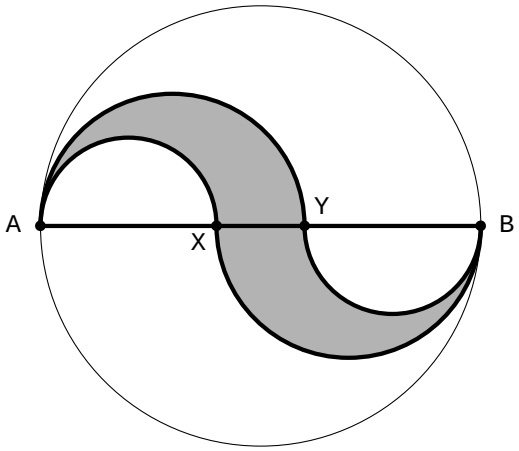
11) Consider the operation $\oplus$, where $(A \oplus B) = AB + B - 1$. Find P so that $(2 \oplus 7) \oplus 5 = (6 \oplus 3) \oplus P$.

12) Determine the value of $\sqrt{\dfrac{27^{10}+9^{10}}{9^{11}+27^4}}$. Express your answer in simplest form.

13) Evaluate $\dfrac{1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}-\frac{1}{2^5}+\cdots+\frac{1}{2^{2008}}-\frac{1}{2^{2009}}}{1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\cdots+\frac{1}{2^{2008}}+\frac{1}{2^{2009}}}$. Express your answer as a rational number in lowest terms.

14) When three people are weighed two at a time, the weights are 280, 330 and 350 pounds. How much does the heaviest person weigh?

15) A circle with diameter $\overline{AB}$ has radius $\frac{5}{\sqrt{\pi}}$. The curves AXB and AYB are formed from semicircles of radii $\frac{2}{\sqrt{\pi}}$ and $\frac{3}{\sqrt{\pi}}$ with centers on $\overline{AB}$. Find the area between the curves AXB and AYB.

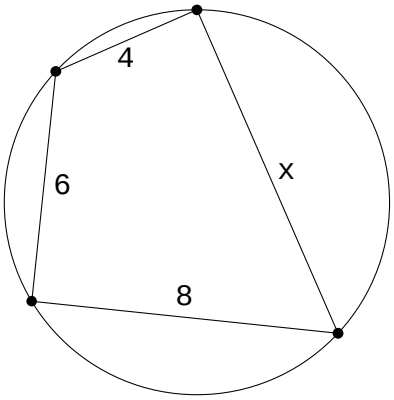


16) Find the integer value of the expression $\log_3(7) \times \left[\log_4(9) + \log_2(3) \right] \times \log_7\left(\frac{1}{4}\right)$.

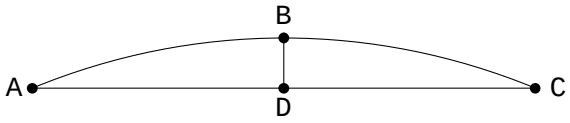
17) In a group of 90 students, there are 48 who play football, 38 who play basketball and 54 who play baseball. Twenty of the 90 students play all three sports. Each student plays at least one sport. How many students play only one sport?

18) If $(1 - i)^{10} = a + b\,i$ where a and b are real numbers and $i^2 = -1$, what is the value of $a + b$?

19) A quadrilateral with sides of length 4, 6, 8 and x is inscribed in a circle of radius 5 (see the figure). What is x?



20) ABC is an arc of a circle, $\overline{AC}$ is a chord of the circle, B is the midpoint of arc ABC, D is the midpoint of chord $\overline{AC}$, AC = 20 and BD = 3. What is the diameter of the circle containing ABC?



21) Let $f(x) = 1 - \frac{1}{x}$. Define the sequence of functions $f_n(x)$ by $f_1(x) = f(x)$ and $f_n(x) = f(f_{n-1}(x))$ for $n \geq 2$. What is $f_{50}(50)$?

22) If S = { 4 , 1 , 12 , x , 7 }, find all real values of x such that the mean of S equals the median of S.

23) How many positive divisors does the integer 1440 have?

24) Define the sequence a_n by $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = a_n + (-1)^n\, a_{n-1}$ for $n \geq 1$. Find all values of n such that $a_n = 144$.

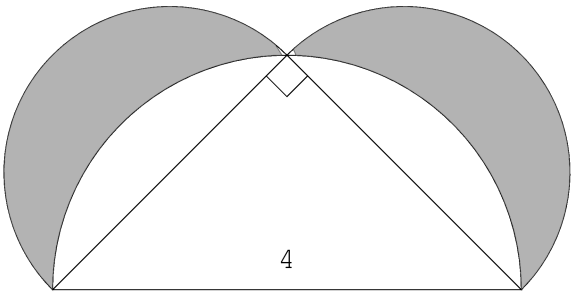
25) Find all real numbers x such that $|\,2x - 1\,| + |\,x - 5\,| = 8$.

26) The corners of a rectangle in the plane are located at the points (1,1), (1,10), (5,10) and (5,1). The line $y = m\,x$ divides the

rectangle into two trapezoids of equal area. Find m.

27) A jar contains 5 different pennies, 6 different nickels, 7 different dimes and 8 different quarters. How many ways can a set of 3 coins be selected so that the total value of the selected coins is less than 15 cents?

28) Semicircles are constructed on the legs and the base of the right isosceles triangle as shown in the diagram. If the length of the base is 4 units, find the sum of the areas of the shaded regions.



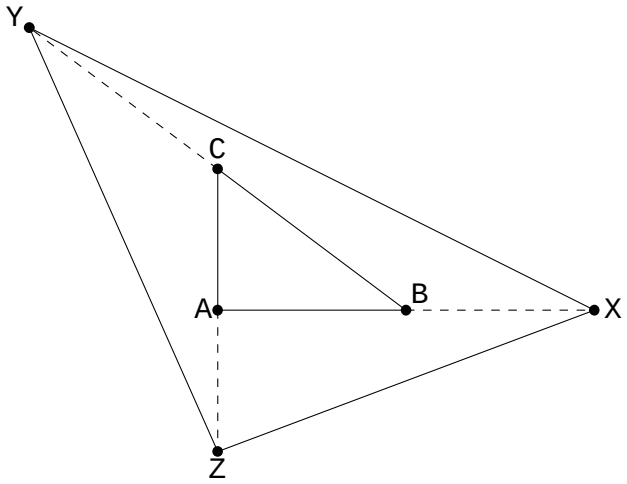
29) If $i^2 = -1$, determine the value of $\left(i + \frac{2}{2+i}\right)^2$. Express your answer in the form $a + b\,i$, where a and b are real numbers.

30) Find the radius of the circle whose locus is the set of all points (x, y) whose distance from $(2, 0)$ is $\sqrt{5}$ times the distance from $(-1, 0)$.

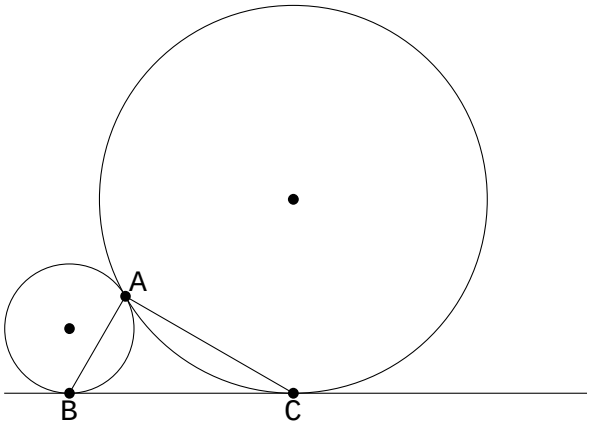
31) Suppose that $x_1, x_2, x_3, \cdots, x_9$ are real numbers with the following property: for $k = 1, 2, 3, \cdots, 9$, the sum of all of the numbers except x_k is k. For example, $x_1 + x_2 + x_3 + x_4 + x_6 + x_7 + x_8 + x_9 = 5$. What is x_3 ?

32) Let $z = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e}}}}}$. If a, b, c, d and e are all either 1 or 2, what is the minimum value of z?

33) ABC is a 3-4-5 right triangle. Side $\overline{AB}$ is extended to point X so that B is the midpoint of $\overline{AX}$, side $\overline{CA}$ is extended to point Z so that A is the midpoint of $\overline{CZ}$ and side $\overline{BC}$ is extended to point Y so that C is the midpoint of $\overline{BY}$. Find the area of triangle XYZ.



34) Let two circles of radii 1 and 3 be tangent at point A and tangent to a common straight line at points B and C. Find $AB^2 + BC^2 + CA^2$.

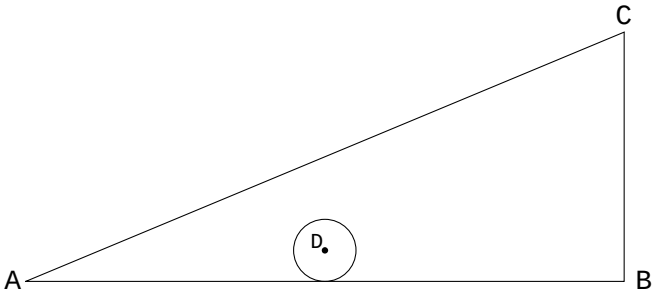


35) How many six-digit positive integers are there in which every digit occurs the same number of times as the value of the digit? An example of such an integer is 133232.

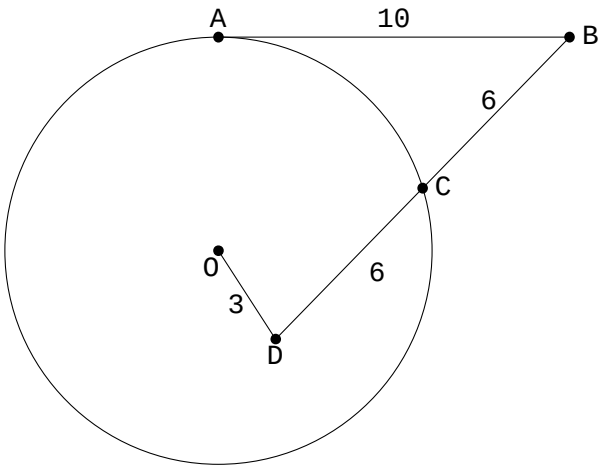
36) An isosceles triangle whose sides are in the ratio 1 : 2 : 2 is inscribed in a circle of radius 1 and a second circle is inscribed in this triangle. What is the radius of the second circle?

37) Determine the value of $\sum_{k=1}^{10} \sum_{j=1}^{k} \left(6j + \frac{1}{k}\right)$.

38) Suppose that the lengths of the sides of a right triangle ABC are 10, 24 and 26. A circle with center D and radius 1 rolls inside the triangle, always tangent to at least one side. How far has D traveled when it first returns to its original position?



39) The line segment $\overline{AB}$ is tangent at A to a circle with center O. Point D is inside the circle and $\overline{DB}$ intersects the circle at C. Given that BC = DC = 6, OD = 3 and AB = 10, find the radius of the circle.



40) Suppose that p, q and r are the distinct roots of $x^3 - 2x^2 + 2x - 3 = 0$. Find the value of $p^3 + q^3 + r^3$.

41) Express $\frac{1}{2\cos(20°)} - 2\sin(50°)$ in simplest form.