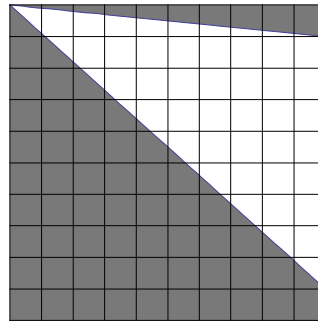


UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
FIFTY-FIRST ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 12, 2008

- 1) At the beginning of the semester, Mr. Kost had 25 students in his class. At the end of the semester, he had 20 students in his class. What was the percent decrease in the number of students in Mr. Kost's class over the semester?
- 2) On a map, a line segment of length $1\frac{1}{4}$ inches represents 40 miles. What distance is represented by a line segment of length 2 inches?
- 3) Simplify $(5^{-2} - 5^{-1})(6^{-2} - 6^{-1})$. Express your answer as a rational number in lowest terms.
- 4) What rational number between $\frac{1}{2}$ and $\frac{2}{3}$ is three times as far from $\frac{1}{2}$ as it is from $\frac{2}{3}$?

- 5) Each small square in the given 10 x 10 square has an area of one square unit.
Find the area of the shaded region.



- 6) If a , b and c are integers such that $a + b + c = 91$ and $abc = 729$, determine the value of $a^2 + b^2 + c^2$.
- 7) The ratio of boys to girls in Mrs. Brown's class is 3 to 2. In Mrs. Smith's class, the ratio of boys to girls is 4 to 3. If there are 30 students in Mrs. Brown's class and 28 students in Mrs. Smith's class, what is the total number of boys in the two classes?
- 8) The sixth grade class at Williston Central School is raffling off a turkey as a fundraising project. If the turkey costs \$22 and the tickets are sold at 75 cents each, how many tickets will have to be sold for the class to make a profit of \$20 ?
- 9) A sweater, originally priced at \$100, is successively discounted by 10%, 15% and 20%. What is the price of the sweater after the third discount?
- 10) An integer $n > 1$ is **abundant** if the sum of its proper divisors (positive integer divisors smaller than n) is greater than n . Find the smallest abundant integer.
- 11) A binary operation \star is defined by $a \star b = a^2 b - b^2 a$. Find $2 \star (3 \star 4)$.

- 12) Larry's retirement portfolio consists of stocks and bonds. In 2000, 30% of the value of his portfolio was in stocks. By 2008, the value of his stocks has doubled, while the value of his bonds has decreased by $\frac{1}{3}$. What fraction of the value of Larry's 2008 retirement portfolio is in bonds? Express your answer as a rational number in lowest terms.
- 13) A day of skiing at Misty Valley costs \$40. However, if you purchase a Misty Pass for \$100, you receive a 30% discount on each day that you ski. Suppose you purchase a Misty Pass at the beginning of the ski season. What is the fewest number of days that you must ski so that your total cost is less than it would have been if you had not purchased a Misty Pass?
- 14) Find all real numbers x such that $\log_3(x) - 2 = \sqrt{\log_3(x^3) - 8}$.
- 15) If $x^4 + x^2 + 1 = 0$, what is the value of $\left(x^2 + \frac{1}{x^2}\right)^3$?
- 16) Let $f(x)$ be a function such that $f(x) + 2f(3-x) = 4x + 5$ for every real number x . What is $f(1)$?
- 17) Let $p(x)$ be a cubic polynomial such that $p(n) = n$ for $n = 0, 1, -1$ and $p(2) = 100$. What is $p(3)$?
- 18) Ship A leaves port at 6:00 a.m. and travels due north at a steady speed of 40 mph. Sometime later, Ship B leaves the same port and travels due east at a steady speed of 36 mph. At 10:00 a.m., the two ships are 200 miles apart. How many minutes after Ship A left port did Ship B leave port?
- 19) What is the largest positive integer n such that $2008!$ is an integer multiple of 15^n ?
- 20) Find all ordered pairs of real numbers (x, y) satisfying both of the equations $3^x 9^y = 81$ and $\frac{2^x}{8^y} = \frac{1}{128}$.
- 21) A sequence $\{a_n\}$ is defined by $a_0 = 3$, $a_1 = 6$, and $a_n = \frac{a_{n-1}}{a_{n-2}}$ for $n \geq 2$. Find a_{2008} .
- 22) A cube is inscribed in a sphere of radius 1. Find the surface area of the cube.
- 23) Find all ordered pairs of real numbers (a, b) such that $\begin{cases} \sqrt{a+b} - \sqrt{a} = 1 \\ \frac{1}{2}b^2 - 4a = 1 \end{cases}$.
- 24) Express $\tan\left(\arctan \frac{1}{2} + \arctan \frac{1}{3} + \arctan \frac{1}{4} + \arctan \frac{1}{5}\right)$ as a rational number in lowest terms.
- 25) Find the number of positive integers $n < 2008$ such that $\binom{2008}{n}$ is divisible by $\binom{2007}{n}$ where $\binom{k}{j} = \frac{k!}{j!(k-j)!}$.
- 26) In the next football season, the Flounders will play 16 games and the outcomes of the games will be recorded as a 16-letter sequence of W's and L's (a W for each win and an L for each loss; there will be no ties). How many such sequences correspond to winning exactly six games with no consecutive wins?
- 27) If $f(x) = ax^4 + bx^2 + 3x + 7$ and $f(-4) = 2008$, determine the value of $f(4)$.
- 28) Find the area of the region in the xy plane consisting of all points (x, y) whose coordinates satisfy the inequalities $|y| - |x| \leq 2$ and $|x| \leq 3$.

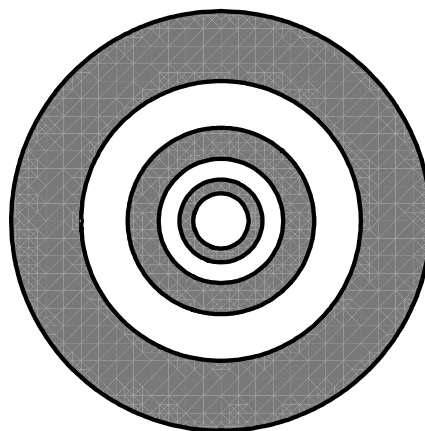
29) Suppose that $\frac{6}{7} = \frac{d_2}{2!} + \frac{d_3}{3!} + \frac{d_4}{4!} + \frac{d_5}{5!} + \frac{d_6}{6!} + \frac{d_7}{7!}$, where each d_j is an integer such that $0 \leq d_j < j$.

Find $d_2 + d_3 + d_4 + d_5 + d_6 + d_7$.

30) Let θ be an acute angle such that $8\cos(2\theta) + 8\sec(2\theta) = 65$. Find the value of $\cos(\theta)$.

31) Let C_n be a circle of radius $\left(\frac{2}{3}\right)^n$ centered at the origin, for $n = 0, 1, 2, \dots$ and let A_n be the area of the region that is inside circle C_{2n} and outside circle C_{2n+1} , for $n = 0, 1, 2, \dots$. The first three regions are shown (shaded) in the figure.

Find $\sum_{n=0}^{\infty} A_n$.



32) In the plane, point A has coordinates $(0, -2)$ and point B has coordinates $(0, 2)$. The region S is the set of points inside the square with vertices at $(-5, \pm 5)$ and $(5, \pm 5)$, and the region T consists of all points X in the plane such that the triangle ABX is obtuse. Find the area of the region common to S and T .

33) A standard die with sides labeled 1 through 6 is rolled three times. Find the probability that the product of the three numbers rolled is divisible by 8. Express your answer as a rational number in lowest terms.

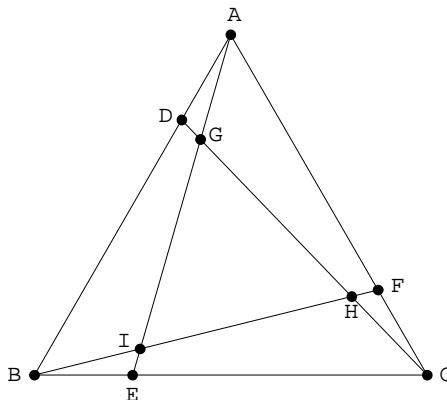
34) Suppose that a, b, c are real numbers such that $\begin{cases} a + b + c = 10 \\ a^2 + b^2 + c^2 = 50 \end{cases}$. Find the maximum value of a .

35) Let r_1, r_2, r_3, r_4, r_5 be the five roots of the equation $x^5 - 6x^4 + 6x^3 + 7x - 5 = 0$.

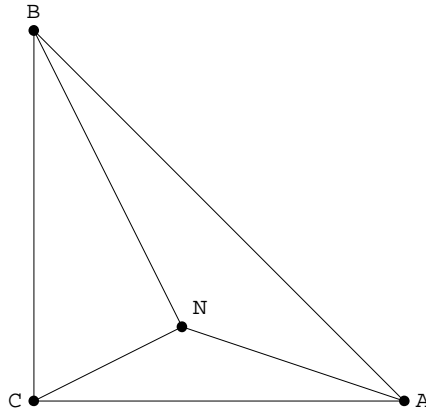
Determine the value of $\sum_{i=1}^5 \sum_{j=1}^5 (r_i - r_j)^2$.

36) Jack and Doug run two races on a 200 yard track. For the first race, Doug lets Jack start 8 yards ahead and also gives him a two second head start. Doug wins this race by 2 seconds. For the second race, Doug lets Jack start 16 yards ahead and also gives him a 5 second head start. Doug loses this race by 20 yards. How many seconds does it take Doug to run the 200 yards?

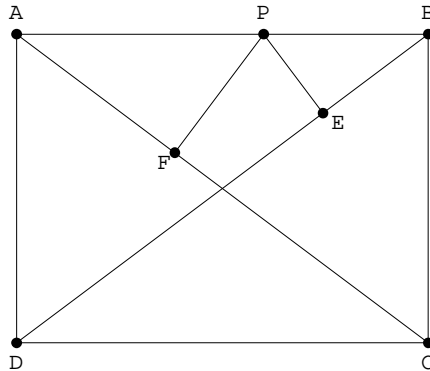
37) ABC is an equilateral triangle of side length 1. Points D, E and F are chosen so that $AD = BE = CF = \frac{1}{4}$. Lines $\overline{CD}, \overline{AE}$ and \overline{BF} are drawn. These lines intersect at points G, H and I . Find the ratio of the area of $\triangle GHI$ to the area of $\triangle ABC$. Express your answer as a rational number in lowest terms.



- 38) ABC is an isosceles right triangle with $CB = CA = 1$. Point N is chosen inside the triangle so that $\angle NBC = \angle NCA = \angle NAB$. Find the area of $\triangle NBC$.



- 39) $ABCD$ is a rectangle with $AB = 4$ and $AD = 3$. Let P be any point on side \overline{AB} . Let E be the point on diagonal \overline{DB} such that $\overline{PE} \perp \overline{DB}$. Let F be the point on diagonal \overline{AC} such that $\overline{PF} \perp \overline{AC}$. Find $PE + PF$.



- 40) If $a + b + c = 3$, $ab + ac + bc = 4$ and $abc = 5$, find $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

- 41) Lines l_1 and l_2 intersect in the plane as shown. Circles A and B , each of radius 1 cm, are drawn tangent to l_1 and l_2 as shown. Circle C is then constructed tangent to l_1 , l_2 , and A (externally), and circle D is constructed tangent to l_1 , l_2 , and B (externally), as shown. If the radius of circle C is 4 cm, find the radius of circle D , in cm.

