UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
FORTY-NINNTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 9, 2006

1) Express $\dfrac{\frac{1}{2}\bullet\frac{3}{4}-\frac{5}{6}+\frac{7}{8}}{\frac{1}{2}+\frac{3}{4}-\frac{5}{6}\bullet\frac{7}{8}}$ as a rational number in lowest terms.

Solution 1

$$\frac{\frac{1}{2}\bullet\frac{3}{4}-\frac{5}{6}+\frac{7}{8}}{\frac{1}{2}+\frac{3}{4}-\frac{5}{6}\bullet\frac{7}{8}} = \frac{\frac{3}{8}-\frac{5}{6}+\frac{7}{8}}{\frac{1}{2}+\frac{3}{4}-\frac{35}{48}}\bullet\frac{48}{48} = \frac{18-40+42}{24+36-35} = \frac{20}{25} = \mathbf{\frac{4}{5}}$$

2) Express $\dfrac{2^{10}-2^{12}}{8^{14/3}}$ as a rational number in lowest terms.

Solution 2

$$\frac{2^{10}-2^{12}}{8^{14/3}} = \frac{2^{10}(1-2^2)}{2^{14}} = -\frac{3}{2^4} = \mathbf{-\frac{3}{16}}$$

3) Using the digits 1, 2, 4, 5, 6, 7 without repetition, how many six–digit numbers can be formed that are divisible by 25 ?

Solution 3

Using the digits 1, 2, 4, 5, 6, 7 to form a six digit number divisible by 25, the last two digits must be 25 or 75. The remaining 4 digits can be arranged in any of 4! = 24 ways. Thus the desired number is 2(24) = **48**.

4) If $x = z^{11}$, then solve $(xz)^{2y} = z^5$ for y. Express your answer as a rational number in lowest terms.

Solution 4

$$x = z^{11} \text{ and } (xy)^{2y} = z^5 \implies (z^{11}z)^{2y} = z^5 \implies z^{24y} = z^5 \implies 24y = 5 \implies y = \mathbf{\frac{5}{24}}$$

5) Express $\dfrac{(50!)^2-(49!)^2}{(50!)^2+(49!)^2}$ as a rational number in lowest terms.

Solution 5

$$\frac{(50!)^2-(49!)^2}{(50!)^2+(49!)^2} = \frac{(49!)^2(50^2-1)}{(49!)^2(50^2+1)} = \frac{2500-1}{2500+1} = \mathbf{\frac{2499}{2501}}$$

6) Solve the equation $\dfrac{1}{y+2}-\dfrac{4}{2y+1}+\dfrac{3y-4}{2y^2+5y+2} = 0$ for y.

Solution 6

$$\frac{1}{y+2}-\frac{4}{2y+1}+\frac{3y-4}{2y^2+5y+2} = 0$$

$$\frac{1}{y+2}-\frac{4}{2y+1}+\frac{3y-4}{(2y+1)(y+2)} = 0$$

$2y + 1 - 4(y + 2) + 3y - 4 = 0$

$2y + 1 - 4y - 8 + 3y - 4 = 0$

$y = \mathbf{11}$

7) Find the exact value of $\log_2(40) + \log_3(27) - \log_2(5)$. Express your answer as a rational number in lowest terms.

Solution 7

$$\log_2(40) + \log_3(27) - \log_2(5) = \log_2(2^3 \bullet 5) + \log_3(3^3) - \log_2(5)$$

$$= \log_2(2^3) + \log_2(5) + 3 - \log_2(5)$$

$$= 3 + \log_2(5) + 3 - \log_2(5) = \mathbf{6}$$

8) Karla sells some eggs to Doug, Larry and Jack . Karla first sells half the eggs plus half an egg to Doug, then sells half the remaining eggs plus half an egg to Larry and finally sells half the remaining eggs plus half an egg to Jack. At the end of the 3 sales, Karla is out of eggs. The strange thing is that Karla never had to break an egg. How many eggs did Karla begin with?

Solution 8

Start with E eggs.

Doug gets $\frac{1}{2}E + \frac{1}{2}$ leaving $E - \frac{1}{2}E - \frac{1}{2} = \frac{1}{2}E - \frac{1}{2}$

Larry gets $\frac{1}{2}(\frac{1}{2}E - \frac{1}{2}) + \frac{1}{2}$ leaving $\frac{1}{2}E - \frac{1}{2} - \frac{1}{4}E - \frac{1}{4} = \frac{1}{4}E - \frac{3}{4}$

Jack gets $\frac{1}{2}(\frac{1}{4}E - \frac{3}{4}) + \frac{1}{2}$ leaving $\frac{1}{4}E - \frac{3}{4} - \frac{1}{8}E - \frac{1}{8} = \frac{1}{8}E - \frac{7}{8} = 0 \implies E = \mathbf{7}$

9) What is the smallest real number k whose distance from -1 is equal to twice its distance from 3?

Solution 9

The given condition leads to $|k+1| = 2|k-3|$

$k \le -1$	$-1 \le k \le 3$	$3 \le k$
$-k-1 = -2k+6$	$k+1 = -2k+6$	$k+1 = 2k-6$
$k = 7$ (can't be)	$k = \frac{5}{3}$	$k = 7$

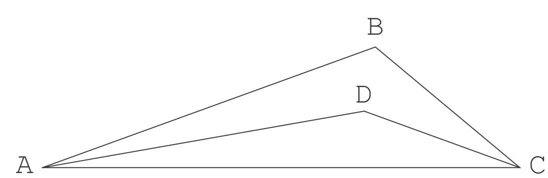
Smallest k = $\mathbf{\frac{5}{3}}$

10) The base 3 representation of an integer n is 21121221211212212112. What is the leading (i.e. left–most) digit in the base 9 representation of n?

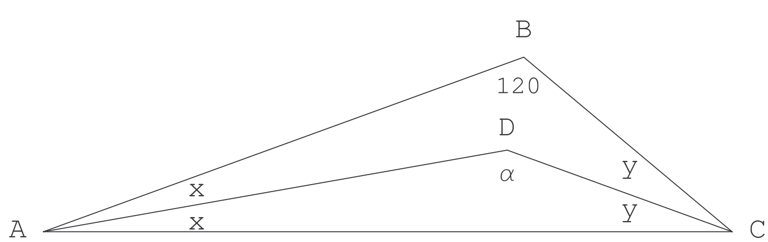
Solution 10

Since $9 = 3^2$ each base 9 digit corresponds to a pair of base 3 digits and the total number of digits in the given string is even, the left-most digit in base 9 is $21_3 = 2 \cdot 3 + 1 = \mathbf{7}$

11) In triangle ABC, the measure of $\angle$ABC is 120°. Point D is chosen in the triangle so that line DA bisects $\angle$BAC and line DC bisects $\angle$BCA. Find the degree measure of $\angle$ADC.



Solution 11



From triangle ABC $2x + 2y + 120 = 180$ (1)

From triangle ADC $x + y + \alpha = 180$ (2)

Multiplying (2) by 2,
$2x + 2y + 2\alpha = 360$ (3)

(3) – (1) $2\alpha - 120 = 180$
$2\alpha = 300$
$\mathbf{\alpha = 150}$

12) A messenger pigeon can fly at a constant speed of 25 miles per hour in a wind–free environment. What is its average speed on a round trip where it faces a 5 mile per hour headwind on the way out and a 5 mile per hour tailwind on the return leg?

Solution 12

Let v_p = velocity of the pigeon = 25 mph, v_w = velocity with the wind = 25 + 5 = 30 mph and v_a = velocity against the wind = 25 – 5 = 20 mph.

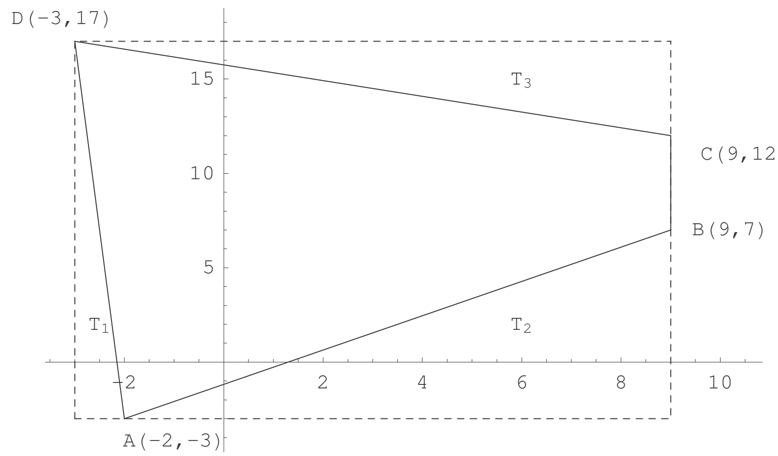
If d is the one way distance and v the average velocity, since velocity = $\frac{\text{distance}}{\text{time}}$ and time = $\frac{\text{distance}}{\text{velocity}}$,

$$v = \frac{2d}{\frac{d}{30}+\frac{d}{20}} = \frac{2}{\frac{1}{30}+\frac{1}{20}} = \frac{2(60)}{2+3} = \frac{120}{5} = \mathbf{24}$$

13) What is the area of the convex quadrilateral whose vertices are $(9, 7)$, $(-2, -3)$, $(-3, 17)$ and $(9, 12)$?

Solution 13

If A_i is the area of triangle T_i, then the area of ABCD = area of rectangle – $A_1 - A_2 - A_3$ = $20 \cdot 12 - \frac{1}{2} \cdot 20(1) - \frac{1}{2} \cdot 10(11) - \frac{1}{2} \cdot 5(12) = \mathbf{145}$



14) Find all real solutions of $4^x + 4^{x+1} = 160$.

Solution 14

$4^x + 4^{x+1} = 160$

$4^x(1+4) = 160$

$4^x = \frac{160}{5}$

$4^x = 32 \Longrightarrow 2^{2x} = 2^5 \Longrightarrow 2x = 5 \Longrightarrow x = \mathbf{\frac{5}{2}}$

15) At what fraction of an hour after 3 o'clock are the minute and hour hands of a twelve–hour clock pointing in the same direction?

Solution 15

Let m_θ be the angle from the vertical of the minute hand and h_θ be the angle from the vertical of the hour hand.

At 3 oclock $m_\theta = 0$ and $h_\theta = 90°$.

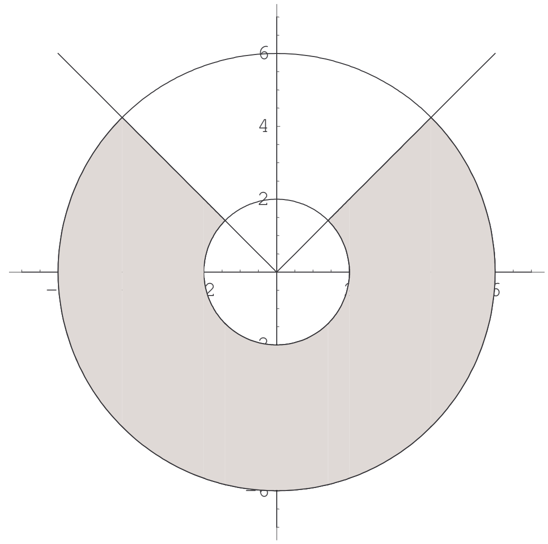
At t minutes after 3 oclock $m_\theta = \frac{360}{60}t$ and $h_\theta = \frac{360}{12(60)}t + 90 = \frac{1}{2}t + 90$

$6t = \frac{1}{2}t + 90 \Longrightarrow \frac{11t}{2} = 90 \Longrightarrow t = \frac{180}{11}$ min $= \frac{\frac{180}{11}}{60}$ hr $= \mathbf{\frac{3}{11}}$ **hr**

16) Find the area of the region of the plane consisting of all points whose coordinates (x, y) satisfy the conditions $4 \le x^2 + y^2 \le 36$ and $y \le |x|$.

Solution 16

The inequalities $4 \le x^2 + y^2 \le 36$ describe the region between circles of radii 2 and 6. $y \le |x|$ describes the region below y = $|x|$. Thus the indicated region can be viewed as the difference in the sectors of two circles. The area of a circular sector of radius r and central angle θ is $\frac{1}{2} r^2\theta$. The indicated are is $\frac{1}{2} \cdot 6^2 \cdot \frac{3\pi}{2} - \frac{1}{2} \cdot 2^2 \cdot \frac{3\pi}{2} = \frac{1}{2} \cdot \frac{3\pi}{2}(36-4) = \frac{3\pi}{4} \cdot 32 =$ **24π**



17) If a, b and c are positive real numbers such that $\log_b(a) = \frac{1}{3}$ and $\log_a(c) = 4$, find the value of $\log_c\left(\frac{a\,b^2}{\sqrt{c}}\right)$.

Express your answer as a rational number in lowest terms.

Solution 17

$\log_b(a) = \frac{1}{3}$ and $\log_a(c) = 4$, find the value of $\log_c\left(\frac{a\,b^2}{\sqrt{c}}\right)$

$\log_a(c) = 4 \implies \log_c(a) = \frac{1}{4} \implies c^{\frac{1}{4}} = a$

$\log_b(a) = \frac{1}{3} \implies \log_a(b) = 3 \implies a^3 = b$

$c^{\frac{1}{4}} = a$ and $a^3 = b \implies c^{\frac{3}{4}} = b \implies \log_c(b) = \frac{3}{4}$

$\log_c\left(\frac{a\,b^2}{\sqrt{c}}\right) = \log_c(a) + 2\log_c(b) - \frac{1}{2}\log_c(c) = \frac{1}{4} + 2 \cdot \frac{3}{4} - \frac{1}{2} =$ **$\frac{5}{4}$**

18) If m and n are positive integers, let $F(m, n) = \frac{L(m,n)}{G(m,n)}$, where $L(m, n)$ is the least common multiple of m and n and $G(m, n)$ is the greatest common divisor of m and n. Find $F(1400, 1760)$.

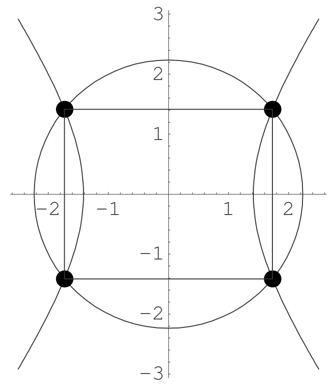
Solution 18

$1400 = 2^3 \cdot 5^2 \cdot 7$ and $1760 = 2^5 \cdot 5 \cdot 11$

$\text{LCM}(1400, 1760) = 2^5 \cdot 5^2 \cdot 7 \cdot 11$ and $\text{GCD}(1400, 1760) = 2^3 \cdot 5$

$\frac{\text{LCM}(1400, 1760)}{\text{GCD}(1400, 1760)} = \frac{2^5 \cdot 5^2 \cdot 7 \cdot 11}{2^3 \cdot 5} = 2^2 \cdot 5 \cdot 7 \cdot 11 =$ **1540**

19) The curves corresponding to the equations $x^2 + y^2 = 5$ and $2x^2 - y^2 = 4$ intersect at four points. These four points are the vertices of a rectangle. Find the area of this rectangle.

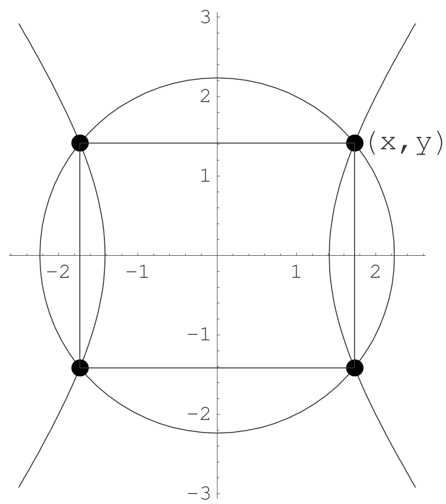


Solution 19

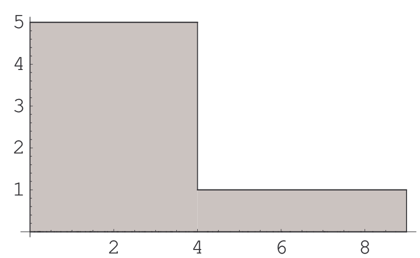
$x^2 + y^2 = 5 \implies y^2 = 5 - x^2$ substituting into $2x^2 - y^2 = 4 \implies 2x^2 - (5 - x^2) = 4 \implies 3x^2 = 9 \implies x = \pm\sqrt{3}$

$x^2 + y^2 = 5$ and $x = \pm\sqrt{3} \implies y = \pm\sqrt{2}$

The area of the rectangle = $2x \cdot 2y = 2\sqrt{3} \cdot 2\sqrt{2} = \mathbf{4\sqrt{6}}$



20) As shown in the diagram, region R in the plane has vertices at (0 , 0), (0 , 5), (4 , 5), (4 , 1), (9 , 1) and (9 , 0). There is a straight line $y = mx$ that partitions R into two subregions of equal area. Find m.



Solution 20

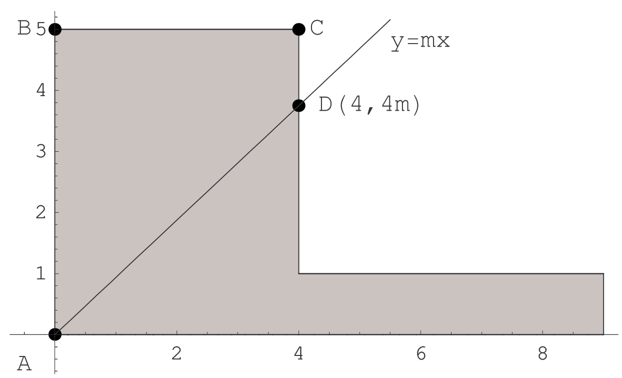
By adding rectangles the total shaded area is 25.

The area of the trapezoid ABCD = $\frac{1}{2}(5 + 5 - 4m)(4) = \frac{25}{2}$

$10 - 4m = \frac{25}{4}$

$4m = 10 - \frac{25}{4} = \frac{15}{4}$

$m = \mathbf{\frac{15}{16}}$



21) Tom and Doug begin their new jobs on the same day. Tom's schedule is 4 work days followed by 2 rest days and Doug's schedule

is 7 work days followed by 3 rest days. In any period of 600 consecutive days, on how many of these days will Tom and Doug have the same rest day?

Solution 21

Tom's schedule repeats every 6 days and Doug's schedule repeats every 10 days. So both will repeat together every 30 days. Letting W represent a work day and R a rest day:

Tom	W	W	W	W	R	R	W	W	W	W	R	R	W	W	W	W	R	**R**	W	W	W	W	R	R	W	W	W	W	**R**	**R**
Doug	W	W	W	W	W	W	W	R	R	R	W	W	W	W	W	W	W	**R**	R	R	W	W	W	W	W	W	W	R	**R**	**R**

Thus for every 30 days they have the same rest day 3 times, so for 600 days the have the same rest day 20(3) = **60 days**.

22) In a convex polygon, the degree measures of the interior angles form an arithmetic progression. If the smallest angle is 159 ° and
the largest angle is 177 °, how many sides does the polygon have?

Solution 22

Let n be the number of sides in the polygon. The sum of the angles of the polygon is $(n-2)\cdot 180$.

Since the angles are in arithmetic progression the sum of the angles is

$$159 + (159 + d) + (159 + 2d) + \cdot\cdot\cdot + (159 + (n-1)d) = 159n + (1 + 2 + + \cdot\cdot\cdot\ n - 1)d$$

$$= 159n + \frac{(n-1)\cdot n}{2} d$$

The largest angle = 177 $\Longrightarrow$ $159 + (n-1)d = 177$ $\Longrightarrow$ $(n-1)d = 177 - 159 = 18$

Then $\quad 159n + \frac{(n-1)\cdot n}{2} d = (n-2)\cdot 180$

Then $\quad 159n + \frac{n}{2} 18 = (n-2)\cdot 180$

$159n + 9n = 180n - 360$

$12n = 360 \quad \Longrightarrow$ n = **30**

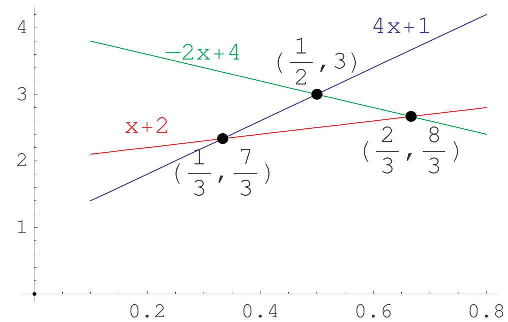
23) For each real number x, let g(x) be the minimum value of the numbers $4x + 1$, $x + 2$ and $-2x + 4$. What is the maximum value of g(x) ?

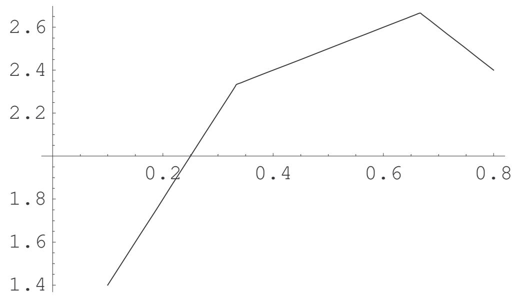
Solution 23

By finding the points of intersection of the pairs of lines and plotting the lines (see below) we find:

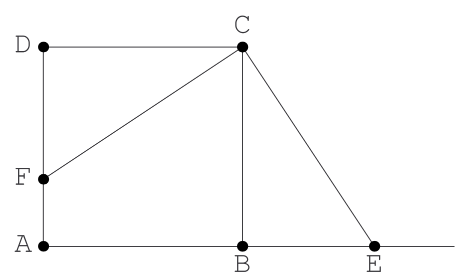
$$g(x) = \begin{cases} 4x + 1 & x \le \frac{1}{3} \\ x + 2 & \frac{1}{3} \le x \le \frac{2}{3} \\ -2x + 4 & x \ge \frac{2}{3} \end{cases}$$

Thus we can see that max(g(x)) = $\mathbf{\frac{8}{3}}$



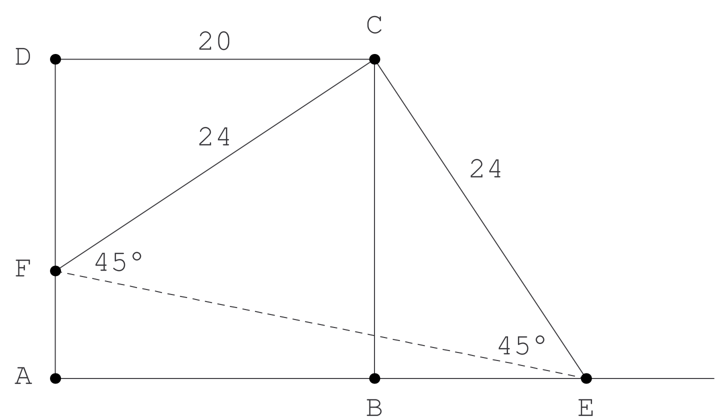


24) In square ABCD, F is a point on side AD and E is a point on the extension of side AB such that $\overline{CF}$ and $\overline{CE}$ are perpendicular. If each side of the square has length 20 and the area of Δ CEF is 288 square units, what is the area of Δ AFE ?



Solution 24

Area Δ CEF = 288 $\Longrightarrow$ $\frac{1}{2}\text{CE} \cdot \text{CF} = 288$ $\Longrightarrow$
$\frac{1}{2}\text{CE}^2 = 288$
$\text{CE}^2 = 576 \Longrightarrow \text{CE} = 24$
$\text{DF} = \text{BE} = \sqrt{24^2 - 20^2} = \sqrt{576 - 400} = \sqrt{176}$
$\text{AF} = 20 - \text{DF} = 20 - \sqrt{176}$
$\text{AE} = 20 + \sqrt{176}$
Area Δ AFE $= \frac{1}{2}\text{AF} \cdot \text{AE} = \frac{1}{2}\left(20 - \sqrt{176}\right)\left(20 + \sqrt{176}\right)$
$= \frac{1}{2}(400 - 176) =$ **112**



25) Find the minimum value of $(\sin(x) - \cos(x) - 1)(\sin(x) + \cos(x) - 1)$.

Solution 25

$$(\sin(x) - \cos(x) - 1)(\sin(x) + \cos(x) - 1) = \sin^2(x) + \sin(x)\cos(x) - \sin(x) - \cos(x)\sin(x) - \cos^2(x) + \cos(x) - \sin(x) - \cos(x) + 1$$

$$= \sin^2(x) - \cos^2(x) - 2\sin(x) + 1$$

$$= \sin^2(x) - (1 - \sin^2(x)) - 2\sin(x) + 1$$

$= 2\sin^2(x) - 2\sin(x)$

Let $t = \sin(x) \implies -1 \le t \le 1$

The expression in t is $2t^2 - 2t = 2t(t-1)$

This is a parabola which opens upward with intercepts 0 and 1. Hence its vertex occurs at $t = \frac{1}{2}$.

Thus the minimum value is $2 \cdot \frac{1}{2}(\frac{1}{2} - 1) =$ $\mathbf{-\frac{1}{2}}$

26) Define the operation $\oplus$ by $a \oplus b = \frac{1}{a} + \frac{1}{b}$. Find all values of c such that $(1 \oplus 2) \oplus c = 1 \oplus (2 \oplus c)$.

Solution 26

$a \oplus b = \frac{1}{a} + \frac{1}{b}$. Find all values of c such that $(1 \oplus 2) \oplus c = 1 \oplus (2 \oplus c)$.

$(1 \oplus 2) \oplus c = (1 + \frac{1}{2}) \oplus c = \frac{3}{2} \oplus c = \frac{2}{3} + \frac{1}{c}$

$1 \oplus (2 \oplus c) = 1 \oplus (\frac{1}{2} + \frac{1}{c}) = 1 + \frac{1}{\frac{1}{2} + \frac{1}{c}} = 1 + \frac{2c}{c+2}$

$\frac{2}{3} + \frac{1}{c} = 1 + \frac{2c}{c+2}$

$\frac{1}{c} = \frac{1}{3} + \frac{2c}{c+2}$ Multiply by $3c(c+2)$

$3c + 6 = c^2 + 2c + 6c^2$

$7c^2 - c - 6 = 0 \implies (7c+6)(c-1) = 0 \implies$ $\mathbf{c = -\frac{6}{7}}$ **or 1**

27) Let the sequence $\{a_n\}$ be defined for all integers n by $a_n - (n+1)\,a_{2-n} = (n+3)^2$. Find a_7.

Solution 27

$a_n - (n+1)\,a_{2-n} = (n+3)^2$. Find a_7.

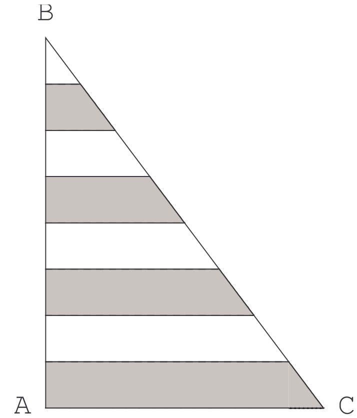
$a_7 - 8\,a_{-5} = 100 \implies a_7 = 100 + 8\,a_{-5}$

$a_{-5} - (-4)\,a_7 = 4 \implies a_{-5} = 4 - 4\,a_7$

$a_7 = 100 + 8\,(4 - 4\,a_7) \implies a_7 = 100 + 32 - 32\,a_7$

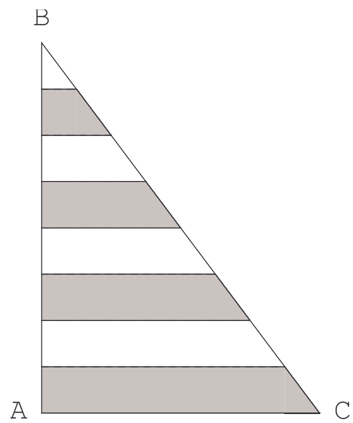
$33\,a_7 = 132 \implies a_7 = \frac{132}{33} =$ $\mathbf{4}$

28) In triangle ABC, AB = 8, AC = 6 and BC = 10. As indicated in the figure, lines are drawn parallel to AC that are one unit apart. Find the area of the shaded region.

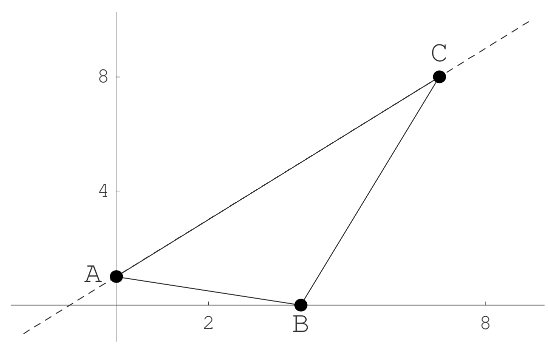


Solution 28

If we number the horizontal lines starting at the bottom as $x_8, x_7 \ldots$, then by similar triangles $\frac{x_\alpha}{\alpha} = \frac{6}{8} = \frac{3}{4}$.
The shaded area is the sum of the areas of trapezoids.
$A = \frac{1}{2}(1)(x_8 + x_7) + \frac{1}{2}(1)(x_6 + x_5) + \ldots$
$A = \frac{1}{2} \cdot \frac{3}{4}((1 + 2 + \ldots + 8)$
$A = \frac{3}{8} \cdot \frac{8(9)}{2} = \frac{27}{2}$

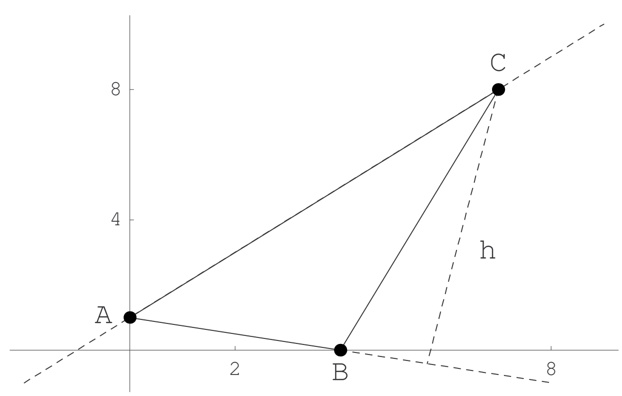


29) Let points A and B have coordinates (0,1) and (4,0) respectively. There is a point C in the first quadrant that lies on the line $y = x + 1$ and such that the area of ΔABC is 20 square units. Find the coordinates of C.



Solution 29

The straight line containing A(0 , 1) and B(4 , 0) can be written as
$x + 4y - 4 = 0$. Let C have coordinates (a , a + 1). Using the formula for the distance from a point to a line h = $\frac{|a + 4(a+1) - 4|}{\sqrt{17}} = \frac{5a}{\sqrt{17}}$
since a is positive. The length of segment AB is $\sqrt{17}$.
Thus the area of the triangle ABC $= \frac{1}{2}\sqrt{17}\,\frac{5a}{\sqrt{17}} = 20$.
Thus $\frac{5}{2}a = 20 \implies a = 8$. So the coordinates of C are **(8 , 9)**.



30) Given positive integers $x_1 \le x_2 \le x_3 \le x_4 \le x_5$ with pairwise sums of 7, 10, 11, 13, 13, 14, 16, 17, 19 and 20, find x_3.

Solution 30

Given positive integers $x_1 \le x_2 \le x_3 \le x_4 \le x_5$ with pairwise sums of 7, 10, 11, 13, 13, 14, 16, 17, 19 and 20, find x_3.

Each of the 5 numbers occurs 4 times in the ten pairwise sums. Thus

$4(x_1 + x_2 + x_3 + x_4 + x_5) = 7 + 10 + 11 + 13 + 13 + 14 + 16 + 17 + 19 + 20 = 140$

$x_1 + x_2 + x_3 + x_4 + x_5 = \frac{140}{4} = 35 \qquad (**)$

Given the relative sizes of the x's, $x_1 + x_2 = 7$ and $x_4 + x_5 = 20$

Substituting into (**) $7 + x_3 + 20 = 35 \implies$ $\mathbf{x_3 = 8}$

31) Determine the coefficient of x^3 in the expansion of $\left(\frac{x^2}{4} + \frac{2}{x}\right)^{12}$.

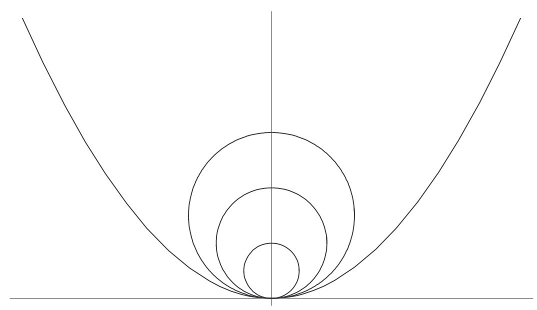
Solution 31

$$\left(\frac{x^2}{4} + \frac{2}{x}\right)^{12} = \sum_{k=0}^{12} \binom{12}{k} \left(\frac{x^2}{4}\right)^{12-k} \left(\frac{2}{x}\right)^{k}$$

To find the appropriate value of k, $x^3 = \frac{x^{24-2k}}{x^k} = x^{24-3k} \implies 3 = 24 - 3k \implies k = 7$

The desired coefficient is $\binom{12}{7} \cdot \frac{2^7}{4^5} = \frac{12!}{7! \cdot 5!} \cdot \frac{2^7}{4^5} = \frac{12!}{7! \cdot 5!} \cdot \frac{2^7}{2^{10}} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2^3} = \mathbf{99}$

32) A parabola has equation $4y = x^2$. There are many circles with centers on the positive y – axis that are tangent to this parabola at $(0, 0)$ and that intersect this parabola only at $(0, 0)$. What is the largest radius of such a circle?



Solution 32

Any such circle has equation $x^2 + (y - a)^2 = a^2$

Using $x^2 = 4y \implies 4y + (y-a)^2 = a^2 \implies 4y + y^2 - 2ay + a^2 = a^2 \implies y^2 + (4 - 2a)y = 0$

$y(y + 4 - 2a) = 0$ Because the curves intersect only when $y = 0$, it follows that $4 - 2a = 0 \implies$ a = **2 .**

33) Find all real values x such that $27^x - 9^{x-1} - 3^{x+1} + \frac{1}{3} = 0$.

Solution 33

$27^x - 9^{x-1} - 3^{x+1} + \frac{1}{3} = 0$

$3^{3x} - \frac{1}{9} 3^{2x} - 3 \cdot 3^x + \frac{1}{3} = 0$

With $\alpha = 3^x$ the equation can be written as:

$\alpha^3 - \frac{1}{9}\alpha^2 - 3\alpha + \frac{1}{3} = 0$

$9\alpha^3 - \alpha^2 - 27\alpha + 9 = 0$

Using the rational root theorem and checking, we find that $\frac{1}{9}$ is a solution. Dividing by $\alpha - \frac{1}{9}$ gives:

$(\alpha - \frac{1}{9})(9\alpha^2 - 27) = 0$

The solutions are $\frac{1}{9}$, $\sqrt{3}$ and $-\sqrt{3}$. Since we are looking for real solutions, the only values are $\frac{1}{9}$ and $\sqrt{3}$.

$3^x = \frac{1}{9} \implies x = \mathbf{-2}$

$3^x = \sqrt{3} \implies x = \mathbf{\frac{1}{2}}$

34) Suppose that $p(x)$ is a polynomial with integer coefficients. The remainder when $p(x)$ is divided by $x - 1$ is 1 and the remainder when $p(x)$ is divided by $x - 4$ is 10. If $r(x)$ is the remainder when $p(x)$ is divided by $(x-1)(x-4)$, find $r(2006)$.

Solution 34

Dividing p(x) by the quadratic $(x-1)(x-4)$ leaves a linear remainder r(x) = ax + b.

$\frac{p(x)}{(x-1)(x-4)} = q(x) + r(x) \implies p(x) = (x-1)(x-4)\,q(x) + (ax + b)$

$p(1) = 1 \implies 1 = a + b$

$p(4) = 10 \implies 10 = 4a + b$

$10 - 1 = 4a + b - a - b \implies 9 = 3a \implies a = 3 \implies b = -2$

$r(x) = 3x - 2 \implies r(2006) = 6018 - 2 = \mathbf{6016}$

35) Suppose that x and y are real numbers such that $\tan(x) + \tan(y) = 42$ and $\cot(x) + \cot(y) = 49$. What is the value of $\tan(x + y)$?

Solution 35

$\tan(x) + \tan(y) = 42$ and $\cot(x) + \cot(y) = 49$

$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$

$\cot(x) + \cot(y) = 49 \implies \frac{1}{\tan(x)} + \frac{1}{\tan(y)} = 49 \implies \frac{\tan(y) + \tan(x)}{\tan(x)\tan(y)} = 49 \implies \tan(x)\tan(y) = \frac{\tan(x) + \tan(y)}{49} = \frac{42}{49} = \frac{6}{7}$

$\tan(x + y) = \frac{42}{1 - \frac{6}{7}} = \frac{42}{\frac{1}{7}} = \mathbf{294}$

36) Suppose that a and b are positive real numbers such that $\log_{27}a + \log_9 b = \frac{7}{2}$ and $\log_{27}b + \log_9 a = \frac{2}{3}$. Determine the value of the product ab.

Solution 36

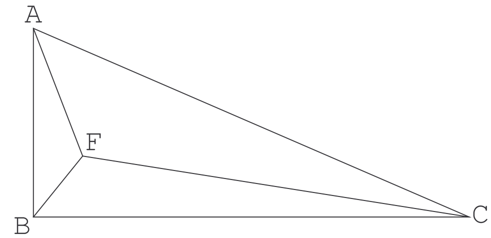
$\log_{27}a + \log_9 b = \frac{7}{2}$
$\log_{27}b + \log_9 a = \frac{2}{3}$

$\frac{1}{3}\log_3 a + \frac{1}{2}\log_3 b = \frac{7}{2}$
$\frac{1}{3}\log_3 b + \frac{1}{2}\log_3 a = \frac{2}{3}$ Add the equations:

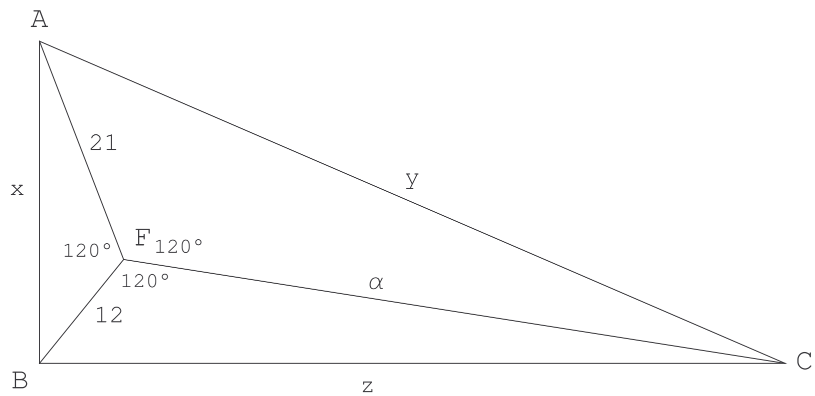
$\frac{1}{3}\log_3(ab) + \frac{1}{2}\log_3(ab) = \frac{7}{2} + \frac{2}{3} = \frac{25}{6}$

$\frac{5}{6}\log_3(ab) = \frac{25}{6} \implies \log_3(ab) = 5 \implies ab = 3^5 =$ **243**

37) In right triangle ABC with right angle at B, an interior point F is located such that FA = 21, FB = 12 and $\angle$ AFB = $\angle$BFC = $\angle$ CFA. Find FC.



Solution 37



Using the Law of Cosines with $\cos(120°) = -\frac{1}{2}$

$x^2 = 21^2 + 12^2 - 2(12)(21)(-\frac{1}{2}) = 21^2 + 12^2 + 12(21)$

$y^2 = 21^2 + \alpha^2 - 2(21)(\alpha)(-\frac{1}{2}) = 21^2 + \alpha^2 + 21\,\alpha$

$z^2 = 12^2 + \alpha^2 - 2(12)(2\alpha)(-\frac{1}{2}) = 12^2 + \alpha^2 + 12\,\alpha$

From the right triangle $y^2 = x^2 + z^2$ Substituting

$21^2 + \alpha^2 + 21\,\alpha = 21^2 + 12^2 + 12(21) + \quad 12^2 + \alpha^2 + 12\,\alpha$

$9\,\alpha = 12(\,2(12) + 21) = 12(45) \implies \alpha = \frac{12\,(45)}{9} =$ **60**

38) Given real numbers a and r, consider the following 20 numbers: $a\,r, a\,r^2, a\,r^3, a\,r^4, \cdots, a\,r^{20}$. If the sum of the 20 numbers is 2006 and the sum of the reciprocals of the 20 numbers is 1003, determine the product of the 20 numbers.

Solution 38

$\frac{a\,r + a\,r^2 + a\,r^3 + a\,r^4 + \cdots + a\,r^{20}}{\frac{1}{a\,r} + \frac{1}{a\,r^2} + \frac{1}{a\,r^3} + \frac{1}{a\,r^4} + \cdots + \frac{1}{a\,r^{20}}.} = \frac{2006}{1003} = 2$ Multiply and divide by $a^2\,r^{21}$

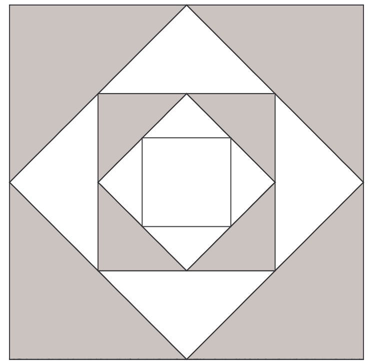
$\frac{a^2\,r^{21}}{a^2\,r^{21}} \; \frac{a\,r + a\,r^2 + a\,r^3 + a\,r^4 + \cdots + a\,r^{20}}{\frac{1}{a\,r} + \frac{1}{a\,r^2} + \frac{1}{a\,r^3} + \frac{1}{a\,r^4} + \cdots + \frac{1}{a\,r^{20}}.} = 2$

$\frac{a^2\,r^{21}\,(\,a\,r + a\,r^2 + a\,r^3 + a\,r^4 + \cdots + a\,r^{20})}{a\,r^{20} + a\,r^{19} + \cdots + a\,r} = 2$

$a^2\,r^{21} = 2$

$\mathrm{a\,r} \cdot \mathrm{a}\,r^2 \cdot \mathrm{a}\,r^3 \cdots \mathrm{a}\,r^{20} = a^{20} \cdot r^{1+2+3+\cdots+20} = a^{20}\,r^{\frac{20(21)}{2}} = a^{20}\,r^{210} = (a^2\,r^{21})^{10} = 2^{10} =$ **1024**

39) Let S_1 be a square of side length 1 unit, let S_2 be the square formed by joining the midpoints of the sides of S_1, let S_3 be the square formed by joining the midpoints of the sides of S_2, etc. Let A_1 be the area inside S_1 and outside S_2, let A_2 the area inside S_3 and outside S_4, etc. (A_1 and A_2 are shown in the figure.) Find $\sum_{k=1}^{\infty} A_k$.



Solution 39

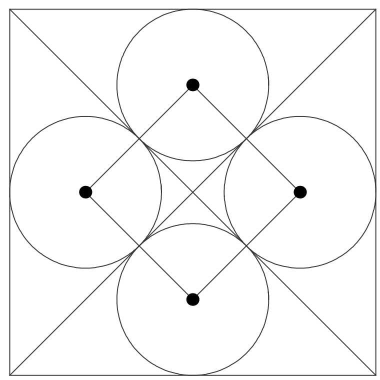
For each pair of squares with sides S_n and S_{n+1}, $(S_{n+1})^2 = (\frac{S_n}{2})^2 + (\frac{S_n}{2})^2 = \frac{1}{2}(S_n)^2$

$(S_1)^2 = 1$, $(S_2)^2 = \frac{1}{2}$, $(S_3)^2 = \frac{1}{4}$, $\cdots$

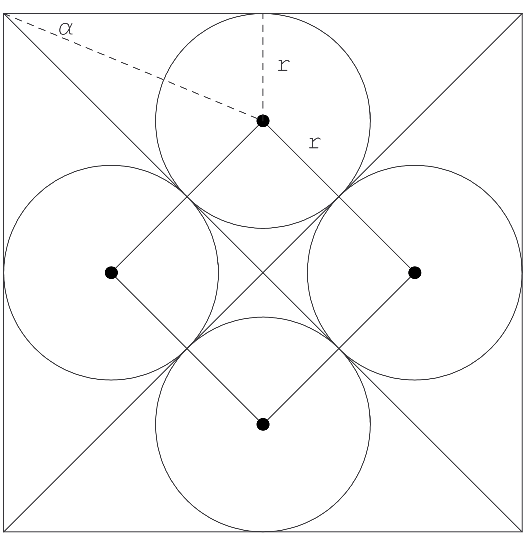
If A_k is the area between consecutive squares $A_1 = 1 - \frac{1}{2} = \frac{1}{2}$, $A_2 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$, $A_3 = \frac{1}{16} - \frac{1}{32} = \frac{1}{32}$, $\cdots$

$A_1 + A_2 + A_3 + \cdots = \frac{1}{2}(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots) = \frac{1}{2}\,\frac{1}{1-\frac{1}{4}} = \frac{1}{2}\cdot\frac{1}{\frac{3}{4}} = \frac{1}{2}\cdot\frac{4}{3} = \mathbf{\frac{2}{3}}$

40) Four congruent circles are inscribed in a square of side length 1, as indicated in the sketch. Find the area of the square formed by joining the centers of the four circles.



Solution 40

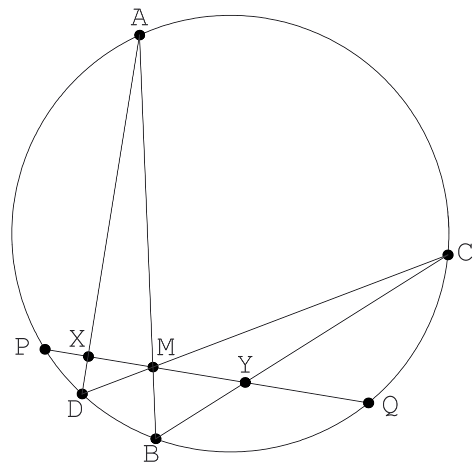


$\alpha = \frac{\pi}{8} \implies \tan(\frac{\pi}{8}) = \frac{r}{\frac{1}{2}} \implies r = \frac{1}{2}\tan(\frac{\pi}{8})$

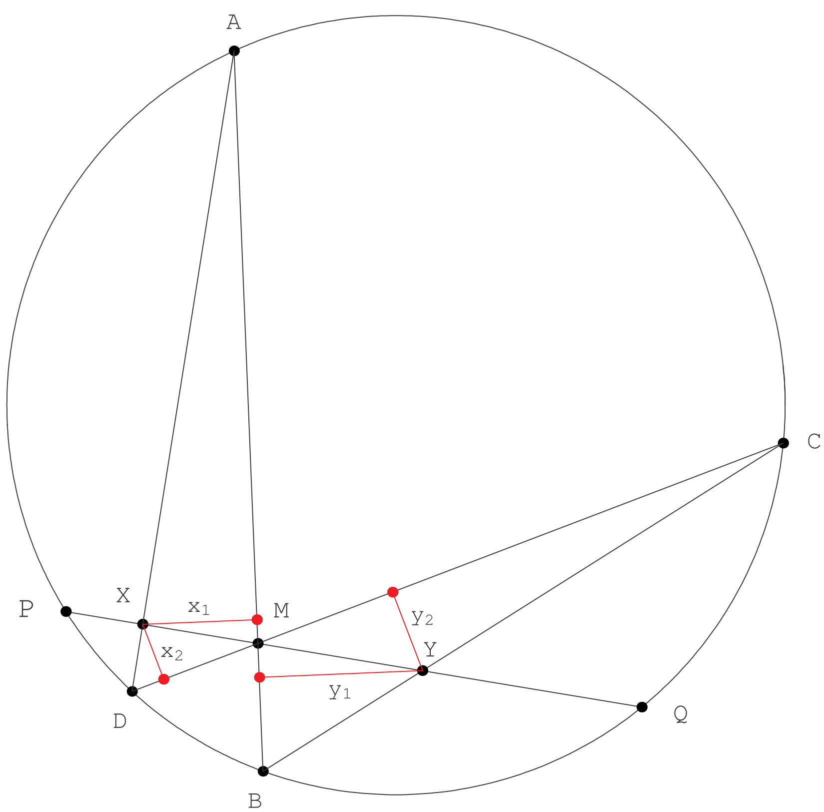
$2r = \tan(\frac{\pi}{8}) = \frac{\sin(\frac{\pi}{8})}{\cos(\frac{\pi}{8})} \implies (2r)^2 = \frac{\sin^2(\frac{\pi}{8})}{\cos^2(\frac{\pi}{8})} = \frac{1-\cos(\frac{\pi}{4})}{1+\cos(\frac{\pi}{4})} = \frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{2+\sqrt{2}}$

$(2r)^2 = \frac{2-\sqrt{2}}{2+\sqrt{2}}\,\frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{4-4\sqrt{2}+2}{4-2} = \frac{6-4\sqrt{2}}{2} = \mathbf{3-2\sqrt{2}}$

41) In a circle of radius 10, M is a point on chord PQ such that PM = 5 and MQ = 10. Chords AB and CD are drawn through M and points X and Y are the respective points of intersection of chords AD and BC with chord PQ. Given that XM = 3, find MY. Express your answer as a rational number in lowest terms.



Solution 41



Note that angle A = angle C and angle B = angle D because each pair of angles is inscribed in the same arc. Draw lines from point X perpendicular to AB and DC and lines from Y perpendicular to DC and AB (as shown in the figure). Using XM = 3 and MY = y, it follows from similar triangles that

$\frac{3}{y} = \frac{x_1}{y_1} = \frac{x_2}{y_2}$, $\frac{x_1}{y_2} = \frac{AX}{CY}$ and $\frac{x_2}{y_1} = \frac{DX}{BY}$. It follows that $\frac{9}{y^2} = \frac{x_1 x_2}{y_1 y_2} = \frac{AX \cdot XD}{CY \cdot YB}$.

By the "Power of a point" theorem $AX \cdot XD = PX \cdot XQ$ and $CY \cdot YB = PY \cdot YQ$.

Then $\frac{9}{y^2} = \frac{PX \cdot XQ}{PY \cdot YQ}$. Using the given information and MY = y, this can be written as:

$\frac{9}{y^2} = \frac{2 \cdot 13}{(5+y)(10-y)}$ Expanding and cross multiplying :

$9(50 + 5y - y^2) = 26\,y^2 \implies 35\,y^2 - 45\,y - 450 = 0 \implies 5\,(7\,y^2 - 9\,y - 90) = 0 \implies 5(7\,y - 30)\,(y + 3) = 0$

Hence, $y = \frac{30}{7}$ or -3. Since y must be positive, **$y = \frac{30}{7}$**.