UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
FORTY-NINTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 9, 2006

1) Express $\dfrac{\frac{1}{2}\bullet\frac{3}{4}-\frac{5}{6}+\frac{7}{8}}{\frac{1}{2}+\frac{3}{4}-\frac{5}{6}\bullet\frac{7}{8}}$ as a rational number in lowest terms.

2) Express $\dfrac{2^{10}-2^{12}}{8^{14/3}}$ as a rational number in lowest terms.

3) Using the digits 1, 2, 4, 5, 6, 7 without repetition, how many six–digit numbers can be formed that are divisible by 25 ?

4) If $x = z^{11}$, then solve $(xz)^{2y} = z^5$ for y. Express your answer as a rational number in lowest terms.

5) Express $\dfrac{(50!)^2-(49!)^2}{(50!)^2+(49!)^2}$ as a rational number in lowest terms.

6) Solve the equation $\dfrac{1}{y+2} - \dfrac{4}{2y+1} + \dfrac{3y-4}{2y^2+5y+2} = 0$ for y.

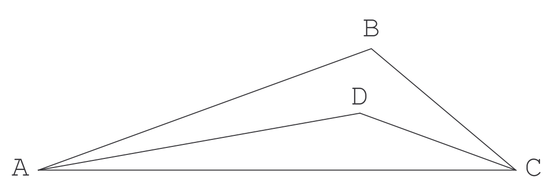
7) Find the exact value of $\log_2(40) + \log_3(27) - \log_2(5)$. Express your answer as a rational number in lowest terms.

8) Karla sells some eggs to Doug, Larry and Jack . Karla first sells half the eggs plus half an egg to Doug, then sells half the remaining eggs plus half an egg to Larry and finally sells half the remaining eggs plus half an egg to Jack. At the end of the 3 sales, Karla is out of eggs. The strange thing is that Karla never had to break an egg. How many eggs did Karla begin with?

9) What is the smallest real number k whose distance from -1 is equal to twice its distance from 3?

10) The base 3 representation of an integer n is 21121221211212212112. What is the leading (i.e. left–most) digit in the base 9 representation of n?

11) In triangle ABC, the measure of $\angle$ABC is 120°. Point D is chosen in the triangle so that line DA bisects $\angle$BAC and line DC bisects $\angle$BCA. Find the degree measure of $\angle$ADC.



12) A messenger pigeon can fly at a constant speed of 25 miles per hour in a wind–free environment. What is its average speed on a round trip where it faces a 5 mile per hour headwind on the way out and a 5 mile per hour tailwind on the return leg?

13) What is the area of the convex quadrilateral whose vertices are $(9, 7)$, $(-2, -3)$, $(-3, 17)$ and $(9, 12)$?

14) Find all real solutions of $4^x + 4^{x+1} = 160$.

15) At what fraction of an hour after 3 o'clock are the minute and hour hands of a twelve–hour clock pointing in the same direction?

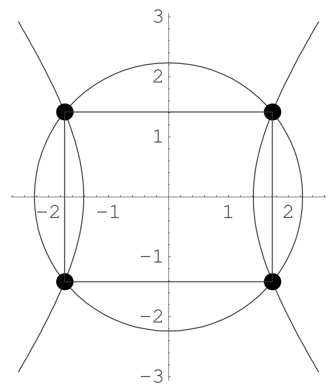
16) Find the area of the region of the plane consisting of all points whose coordinates (x, y) satisfy the conditions $4 \le x^2 + y^2 \le 36$ and $y \le |x|$.

17) If a, b and c are positive real numbers such that $\log_b(a) = \frac{1}{3}$ and $\log_a(c) = 4$, find the value of $\log_c\left(\frac{a b^2}{\sqrt{c}}\right)$.

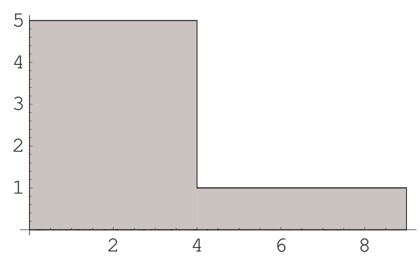
Express your answer as a rational number in lowest terms.

18) If m and n are positive integers, let $F(m, n) = \frac{L(m,n)}{G(m,n)}$, where $L(m, n)$ is the least common multiple of m and n and $G(m, n)$ is the greatest common divisor of m and n. Find $F(1400, 1760)$.

19) The curves corresponding to the equations $x^2 + y^2 = 5$ and $2x^2 - y^2 = 4$ intersect at four points. These four points are the vertices of a rectangle. Find the area of this rectangle.



20) As shown in the diagram, region R in the plane has vertices at (0 , 0), (0 , 5), (4 , 5), (4 , 1), (9 , 1) and (9 , 0). There is a straight line $y = m x$ that partitions R into two subregions of equal area. Find m.



21) Tom and Doug begin their new jobs on the same day. Tom's schedule is 4 work days followed by 2 rest days and Doug's sched-ule

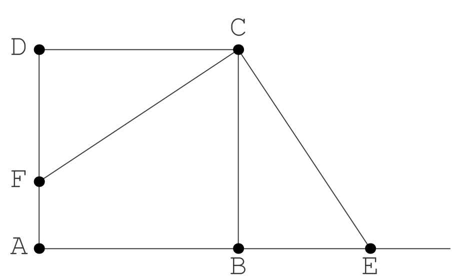
is 7 work days followed by 3 rest days. In any period of 600 consecutive days, on how many of these days will Tom and Doug have the same rest day?

22) In a convex polygon, the degree measures of the interior angles form an arithmetic progression. If the smallest angle is 159 ° and

the largest angle is 177 °, how many sides does the polygon have?

23) For each real number x, let $g(x)$ be the minimum value of the numbers $4x + 1$, $x + 2$ and $-2x + 4$. What is the maximum value of $g(x)$?

24) In square ABCD, F is a point on side AD and E is a point on the extension of side AB such that $\overline{CF}$ and $\overline{CE}$ are perpendicular. If each side of the square has length 20 and the area of Δ CEF is 288 square units, what is the area of Δ AFE ?

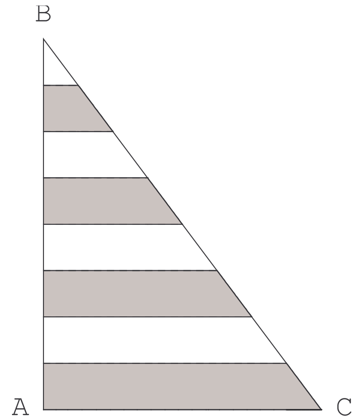


25) Find the minimum value of $(\sin(x) - \cos(x) - 1)(\sin(x) + \cos(x) - 1)$.

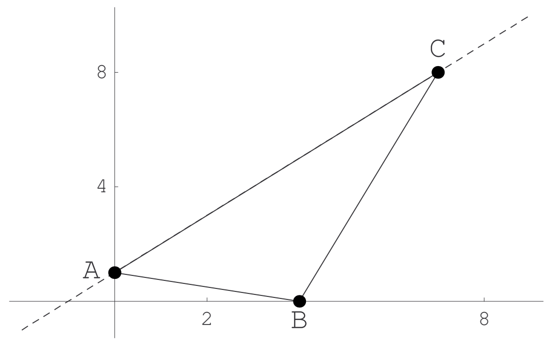
26) Define the operation $\oplus$ by $a \oplus b = \frac{1}{a} + \frac{1}{b}$. Find all values of c such that $(1 \oplus 2) \oplus c = 1 \oplus (2 \oplus c)$.

27) Let the sequence $\{a_n\}$ be defined for all integers n by $a_n - (n+1)\, a_{2-n} = (n+3)^2$. Find a_7.

28) In triangle ABC, AB = 8, AC = 6 and BC = 10. As indicated in the figure, lines are drawn parallel to AC that are one unit apart. Find the area of the shaded region.



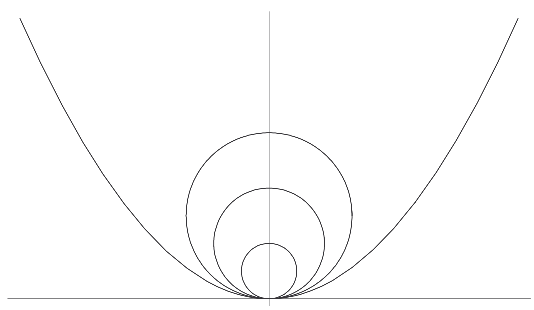
29) Let points A and B have coordinates (0,1) and (4,0) respectively. There is a point C in the first quadrant that lies on the line $y = x + 1$ and such that the area of ΔABC is 20 square units. Find the coordinates of C.



30) Given positive integers $x_1 \le x_2 \le x_3 \le x_4 \le x_5$ with pairwise sums of 7, 10, 11, 13, 13, 14, 16, 17, 19 and 20, find x_3.

31) Determine the coefficient of x^3 in the expansion of $\left(\frac{x^2}{4} + \frac{2}{x}\right)^{12}$.

32) A parabola has equation $4y = x^2$. There are many circles with centers on the positive y – axis that are tangent to this parabola at (0 , 0) and that intersect this parabola only at (0 , 0). What is the largest radius of such a circle?



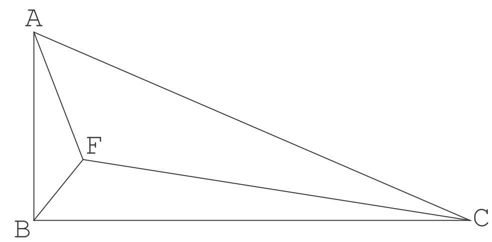
33) Find all real values x such that $27^x - 9^{x-1} - 3^{x+1} + \frac{1}{3} = 0$.

34) Suppose that $p(x)$ is a polynomial with integer coefficients. The remainder when $p(x)$ is divided by $x - 1$ is 1 and the remainder when $p(x)$ is divided by $x - 4$ is 10. If $r(x)$ is the remainder when $p(x)$ is divided by $(x - 1)(x - 4)$, find $r(2006)$.

35) Suppose that x and y are real numbers such that $\tan(x) + \tan(y) = 42$ and $\cot(x) + \cot(y) = 49$. What is the value of $\tan(x + y)$?

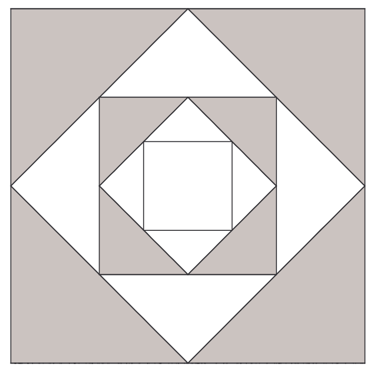
36) Suppose that a and b are positive real numbers such that $\log_{27} a + \log_9 b = \frac{7}{2}$ and $\log_{27} b + \log_9 a = \frac{2}{3}$. Determine the value of the product ab.

37) In right triangle ABC with right angle at B, an interior point F is located such that FA = 21, FB = 12 and $\angle$ AFB = $\angle$BFC = $\angle$ CFA. Find FC.

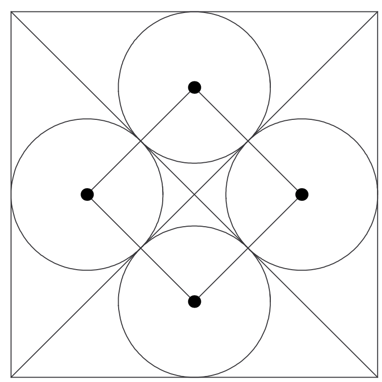


38) Given real numbers a and r, consider the following 20 numbers: $a\,r, a\,r^2, a\,r^3, a\,r^4, \cdots, a\,r^{20}$. If the sum of the 20 numbers is 2006 and the sum of the reciprocals of the 20 numbers is 1003, determine the product of the 20 numbers.

39) Let S_1 be a square of side length 1 unit, let S_2 be the square formed by joining the midpoints of the sides of S_1, let S_3 be the square formed by joining the midpoints of the sides of S_2, etc. Let A_1 be the area inside S_1 and outside S_2, let A_2 the area inside S_3 and outside S_4, etc. (A_1 and A_2 are shown in the figure.) Find $\sum_{k=1}^{\infty} A_k$.



40) Four congruent circles are inscribed in a square of side length 1, as indicated in the sketch. Find the area of the square formed by joining the centers of the four circles.



41) In a circle of radius 10, M is a point on chord PQ such that PM = 5 and MQ = 10. Chords AB and CD are drawn through M and points X and Y are the respective points of intersection of chords AD and BC with chord PQ. Given that XM = 3, find MY. Express your answer as a rational number in lowest terms.