UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
FORTY-EIGHTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 3, 2005

Solution 1

$$\frac{\sqrt{\frac{9}{11}}}{\sqrt{5+\frac{9}{11}}}\ \frac{\sqrt{11}}{\sqrt{11}} = \frac{\sqrt{9}}{\sqrt{55+9}} = \frac{3}{\sqrt{64}} = \mathbf{\frac{3}{8}}$$

Solution 2

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5} = \frac{30+40+45+48}{60} = \mathbf{\frac{163}{60}}$$

Solution 3

$$\frac{99-x}{100-x} = \frac{98}{100}$$

$100(99-x) = 98(100-x)$

$100(99) - 100x = 100(98) - 98x$

$100(99-98) = 2x$

$100 = 2x \implies x = \mathbf{50}$

Solution 4

$A_1 = 3$ and $A_{n+1} = -1 + 2 \cdot A_n$ for $n \geq 1$

$A_2 = 2(3) - 1 = 5$

$A_3 = 2(5) - 1 = 9$

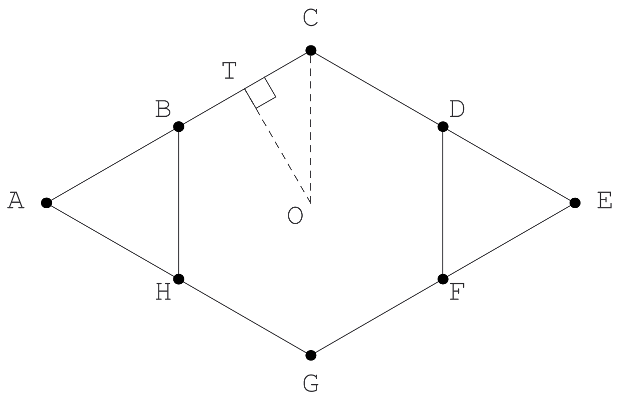
$A_4 = 2(9) - 1 = 17$

$A_5 = 2(17) - 1 = 33$

$A_6 = 2(33) - 1 = 65$

$A_7 = 2(65) - 1 = \mathbf{129}$

Solution 5



Let O be the center of the hexagon. $OC = \frac{1}{2}CG = 5$. Angle $TOC = \frac{\pi}{6} \Longrightarrow CT = \frac{5}{2} \Longrightarrow CB = 5$.

Angles ABH and AHB are each $\frac{\pi}{3}$ $\Longrightarrow$ triangle ABH is equilateral $\Longrightarrow AH = BH = CB = \mathbf{5}$

Solution 6

$$\frac{100! - 99! - 98!}{100! + 99! + 98!} = \frac{98!\,(100\cdot 99 - 99 - 1)}{98!\,(100\cdot 99 + 99 + 1)} = \frac{9800}{1000} = \mathbf{\frac{49}{50}}$$

Solution 7

$$\left\{\begin{array}{rl} 2A + 3G + 5O = 9 & (1)\\ A + 2G + O = 4 & (2)\\ 5A + 4G + 3O = 10 & (3)\end{array}\right.$$

$$\left\{\begin{array}{rr} 2A + 3G + 5O = 9 & (1)\\ -2A - 4G - 2O = -8 & -2\,(2)\\ \hline -G + 3O = 1 & (a)\end{array}\right.$$

$$\left\{\begin{array}{rr} 5A + 4G + 3O = 10 & (3)\\ -5A - 10G - 5O = -8 & -5\,(2)\\ \hline -6G - 2O = -10 & (b)\end{array}\right.$$

$$\left\{\begin{array}{rr} -2G + 6O = 2 & 2\,(a)\\ -18G - 6O = -30 & 3\,(b)\\ \hline -20G = -28 & (c)\end{array}\right.$$

$G = \frac{28}{20} = \frac{7}{5} = \mathbf{\$1.40}$

Solution 8

$$\frac{\text{Number good choices}}{\text{Total number of choices}} = \frac{\binom{8}{3}}{\binom{12}{3}} = \frac{\frac{8!}{3!\cdot 5!}}{\frac{12!}{3!\cdot 9!}} = \frac{\frac{8\cdot 7\cdot 6}{3\cdot 2}}{\frac{12\cdot 11\cdot 10}{3\cdot 2}} = \frac{8\cdot 7\cdot 6}{12\cdot 11\cdot 10} = \mathbf{\frac{14}{55}}$$

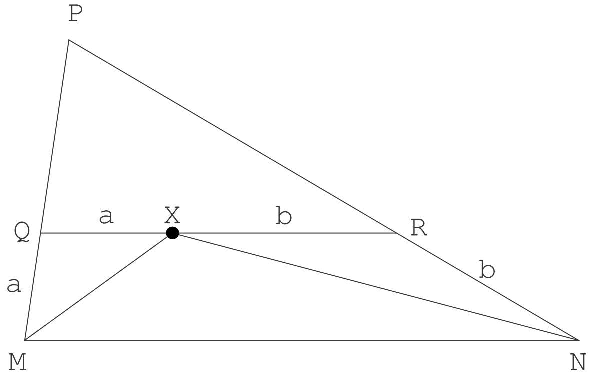
Solution 9

Let $\alpha = \sqrt{7+4\sqrt{3}} + \sqrt{7-4\sqrt{3}}$

$$\alpha^2 = 7+4\sqrt{3} + 2\sqrt{7+4\sqrt{3}} \cdot \sqrt{7-4\sqrt{3}} + 7-4\sqrt{3}$$

$$\alpha^2 = 14 + 2\sqrt{49-48} = 14+2 = 16 \implies \alpha = 4$$

Solution 10



Perimeter Δ PQR = 10 – a + a + b + 17 – b = **27**

Solution 11

$$\frac{1}{2} + \frac{1}{\frac{1}{3} + \frac{1}{\frac{1}{4}+5}} = \frac{1}{2} + \frac{1}{\frac{1}{3} + \frac{1}{\frac{21}{4}}} = \frac{1}{2} + \frac{1}{\frac{1}{3} + \frac{4}{21}} = \frac{1}{2} + \frac{1}{\frac{11}{21}} = \frac{1}{2} + \frac{21}{11} = \frac{11}{22} + \frac{42}{22} = \frac{53}{22}$$

Solution 12

2 + 4 + · · · 1000 – (1 + 3 + · · · + 999) = 2 + 4 + · · · + 1000 – [(1 + 3 + · · · + 999) + (2 + 4 + · · · + 1000) – (2 + 4 + · · · + 1000)]

$$= 2\,(2+4+\cdots+1000) - (1+2+3+\cdots+1000)$$

$$= 2 \cdot 2\,(1+2+\cdots+500) - (1+2+3+\cdots+1000)$$

$$= 4\,\frac{500 \cdot 501}{2} - \frac{1000 \cdot 1001}{2}$$

$$= 2 \cdot 500 \cdot 501 - 500 \cdot 1001 = 500 \cdot (2 \cdot 501 - 1001) = 500 \cdot (1002 - 1001) = 500$$

OR

2 + 4 + · · · 1000 – (1 + 3 + · · · + 999) = (2 – 1) + (4 – 3) + (6 – 5) + · · · + (1000 – 999) = 1 + 1 + 1 + · · · + 1 = **500**

Solution 13

$75 + 75 + d + 75 + 2d + 75 + 3d = 360$

$6d = 360 - 300 = 60 \implies d = 10$

$75 + 3d = 75 + 30 =$ **105**

Solution 14

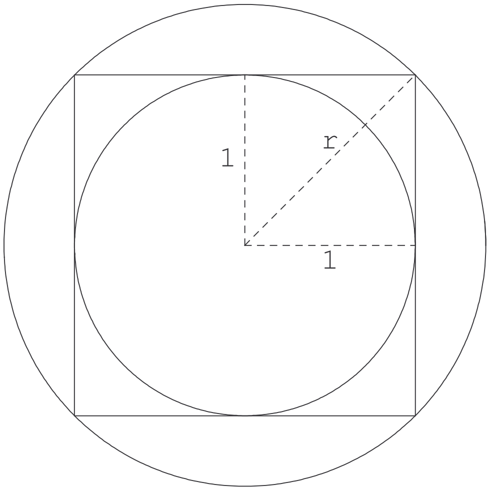
Let L and R be the rate $\left(\frac{\text{eggs}}{\text{day}}\right)$ for the Leghorn hens and 3 New Hampshire Red hens respectively.

$(4L + 3R) \cdot 15 = (3L + 5R) \cdot 12 \implies \frac{R}{L} = \frac{8}{5}$

$L \cdot x = 20 \cdot R$

$x = 20 \cdot \frac{R}{L} = 20 \cdot \frac{8}{5} =$ **32**

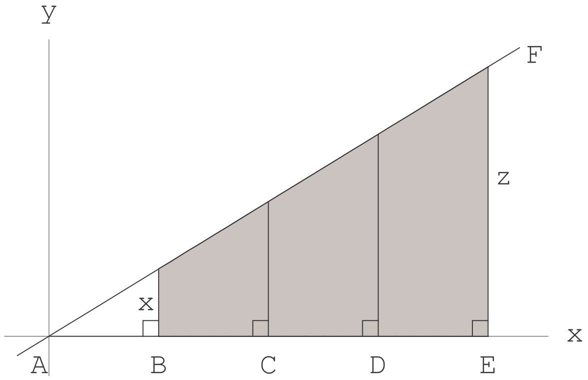
Solution 15



$r^2 = 1 + 1 = 2 \implies r = \sqrt{2}$

$S = 4\pi r^2 = 4\pi(2) =$ **8π**

Solution 16



$\frac{x}{7} = \frac{z}{28} \implies \text{x} = \frac{z}{4}$ By similar triangles

$\frac{1}{2}(\text{x} + \text{z}) \cdot 21 = 18$ Area of trapezoid

$\frac{1}{2}\left(\frac{z}{4} + z\right) \cdot 21 = 18$

$\frac{1}{2}\ \frac{5}{4}\text{z} \cdot 21 = 18$

$\text{z} = \frac{18 \cdot 8}{5 \cdot 21}$

$\frac{z}{28} = \frac{18 \cdot 8}{5 \cdot 21 \cdot 28} = \boldsymbol{\frac{12}{245}}$

Solution 17

$\sin(2x) = 2\sin^2(x)$

$2\sin(x)\cos(x) - 2\sin^2(x) = 0$

$2\sin(x)\,(\cos(x) - \sin(\text{x})) = 0$

$\sin(x) = 0 \implies x = 0, \pi, 2\pi$

$\cos(x) = \sin(x) \implies \tan(x) = 1 \implies x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

Number of solutions = **5**

Solution 18

$1010101_2 = 64 + 16 + 4 + 1 = 85$

$85^2 = 7225$

$7255 = 2(3612) + 1$

$3612 = 2(1806) + 0$

1806 = 2(903) + 0

903 = 2(451) + 1

451 = 2(225) + 1

225 = 2(112) + 1

112 = 2(56) + 0

56 = 2(28) + 0

28 = 2(14) + 0

14 = 2(7) + 0

7 = 2(3) + 1

3 = 2(1) + 1

1 = 2(0) + 1

$7225 = 1110000111001_2$

Number of 1's = **7**

Solution 19

It is easier to count the number of permutations in which no letter is in a position originally occupied by the same letter. In such an arrangement, the A's must occupy the positions originally occupied by the B and two N's. Then place the B in one of the three remaining positions and the positions of the N's are forced. Thus there are 3 such arrangements.

The total number of permutations is $\frac{6!}{3!\,2!} = 60$. So the required probability is $\frac{60-3}{60} = \frac{57}{60} = \mathbf{\frac{19}{20}}$

Solution 20

$y = 6x^2 + 9x + 1$ and $y = 2x + 4$

$2x + 4 = 6x^2 + 9x + 1 \implies 6x^2 + 7x - 3 = 0$

$(3x - 1)(2x + 3) = 0$

x = $\mathbf{\frac{1}{3}}$ **or** $\mathbf{-\frac{3}{2}}$

Solution 21

$$\frac{3}{x^2 + 7x + 12} + \frac{2}{x^2 - 9} = \frac{4}{x^2 + x - 12}$$

$\frac{3}{(x+3)(x+4)} + \frac{2}{(x-3)(x+3)} = \frac{4}{(x+4)(x-3)}$ Multiply by $(x+3)(x-3)(x+4)$

$3(x-3) + 2(x+4) = 4(x+3)$

$3x - 9 + 2x + 8 = 4x + 12$

$\mathbf{x = 13}$

Solution 22

Let r be the smaller radius and R the larger radius.

$2\pi R - 2\pi r = r$ and $\pi R^2 - \pi r^2 = \frac{1}{2\pi} + 2$

From the first equation $R = r\,\frac{1+2\pi}{2\pi}$

Substitute into the second equation

$\pi\,(r\,\frac{1+2\pi}{2\pi})^2 - \pi r^2 = \frac{1}{2\pi} + 2$

$r^2[\frac{(1+2\pi)^2 - 4\pi^2}{4\pi}] = \frac{1+4\pi}{2\pi}$

$r^2[\frac{1+4\pi}{4\pi}] = \frac{1+4\pi}{2\pi}$

$r^2 = 2$

$\mathbf{r = \sqrt{2}}$

Solution 23

Let D = cost of driving the car last year in $\frac{\text{dollars}}{\text{mile}}$. Then 20000 D = 1200 $\Longrightarrow$ D = $\frac{1200}{20000} = \frac{3}{50}$

For this year the cost will be $\frac{4}{5} \cdot 20000 \cdot \frac{7}{5} \cdot \text{D} = \frac{4}{5} \cdot 20000 \cdot \frac{7}{5} \cdot \frac{3}{50} =$ **1344**

Solution 24

Let r be the radius of the smaller circle and θ the central angle in radians.

Area of CEFD = $\frac{1}{2}\,60^2\theta - \frac{1}{2}\,r^2\,\theta = 8 \cdot$ area of A $= 8\,(\frac{1}{2}\,r^2\,\theta)$

$3600 - r^2 = 8\,r^2$

$9\,r^2 = 3600 \Longrightarrow r^2 = 400 \Longrightarrow$ **r = 20**

Solution 25

$xy + 5x + y = 2000$

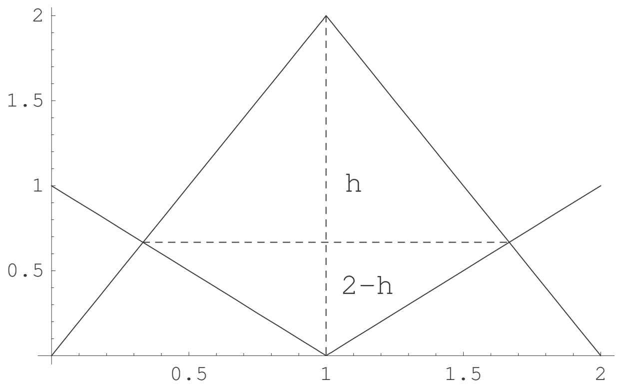
$y(x+1) = 2000 - 5x$

$y = \frac{2000-5x}{x+1}$ Divide

$y = -5 + \frac{2005}{x+1} = -5 + \frac{5(401)}{x+1}$ For this to be an integer $x+1=5$ or $x+1=401 \Longrightarrow x=4$ or $x=400$

If $x=4$, $y=396$. If $x=400$, $y=0$. The positive value of y is **396**.

Solution 26



Find points of intersection.

If x < 1, $1 - x = 2 + 2x - 2 \Longrightarrow x = \frac{1}{3}$ and $y = \frac{2}{3}$

If x > 1, $x - 1 = 2 - 2x + 2 \Longrightarrow x = \frac{5}{3}$ and $y = \frac{2}{3}$

The length of the horizontal dashed line = $\frac{5}{3} - \frac{1}{3} = \frac{4}{3}$

The desired area = sum of the areas of two triangle = $\frac{1}{2} \cdot h \cdot \frac{4}{3} + \frac{1}{2} \cdot (2-h) \cdot \frac{4}{3} = \mathbf{\frac{4}{3}}$

Solution 27

$(1 \oplus 2) \oplus c = 1 \oplus (2 \oplus c)$

$1 \oplus 2 = \frac{1-2}{1+2} = -\frac{1}{3}$

$(1 \oplus 2) \oplus c = \frac{-\frac{1}{3} - c}{-\frac{1}{3} + c} = \frac{1+3c}{1-3c}$

$2 \oplus c = \frac{2-c}{2+c}$

$1 \oplus (2 \oplus c) = \frac{1 - \frac{2-c}{2+c}}{1 + \frac{2-c}{2+c}} = \frac{2+c-2+c}{2+c+2-c} = \frac{c}{2}$

$\frac{1+3c}{1-3c} = \frac{c}{2}$

$2 + 6c = c - 3c^2$

$3c^2 + 5c + 2 = 0$

$(3c + 2)(c + 1) = 0 \implies$ **$c = -\frac{2}{3}$ or -1**

Solution 28

$9! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 = 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot 2 \cdot 3 \cdot 7 \cdot 2^3 \cdot 3^2 = 2^7 \cdot 3^4 \cdot 5 \cdot 7$

To be a perfect square the exponents of the prime factors must all be even. Thus the smallest multiple of 9! which is a perfect square is $2 \cdot 5 \cdot 7 =$ **70**

Solution 29

Let a_n be the n^{th} term of the sequence. Then $S_n = a_1 + a_2 + \cdots + a_n = n^2 + 5n$

For any n $a_n = S_n - S_{n-1} = n^2 + 5n - (n-1)^2 - (n-1) = 2n + 4$

$a_{2005} = 2 \cdot 2005 + 4 =$ **4014**

Solution 30

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{r_2 r_3 + r_1 r_3 + r_1 r_2}{r_1 r_2 r_3}$$

Expanding $(x - r_1)(x - r_2)(x - r_3) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_2 r_3 + r_1 r_3 + r_1 r_2)x + r_1 r_2 r_3$

Comparing with $x^3 + 2x^2 - 6x + 9$, $r_2 r_3 + r_1 r_3 + r_1 r_2 = -6$ and $r_1 r_2 r_3 = -9$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{r_2 r_3 + r_1 r_3 + r_1 r_2}{r_1 r_2 r_3} = \frac{-6}{-9} = \mathbf{\frac{2}{3}}$$

Solution 31

$y = \frac{2}{3}x + 11$ and the circle $x^2 + y^2 - 4x + 10y + 16 = 0$.

Find the center of the circle by completing the square $x^2 + y^2 - 4x + 10y + 16 = x^2 - 4x + 4 + y^2 + 10y + 25 = -16 + 4 + 25$

$(x-2)^2 + (y+5)^2 = 13$ center $= (2, -5)$ radius $= \sqrt{13}$

Given a straight line $ax + by + c = 0$ and a point (α, β), then distance from the point to the line is $\frac{|a \cdot \alpha + b \cdot \beta + c|}{\sqrt{a^2 + b^2}}$.

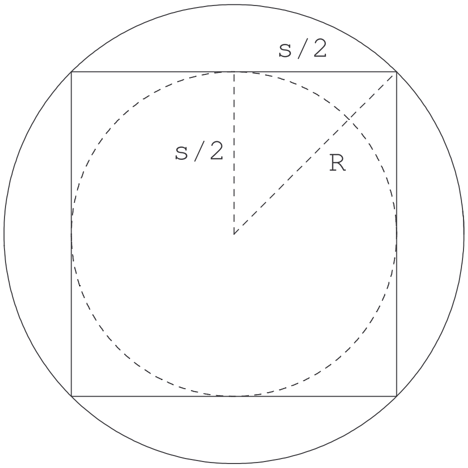
The given line can be written as $2x - 3y + 33 = 0$.

Thus the distance from the center of the circle to the line is $\frac{|2(2) - 3(-5) + 33|}{\sqrt{2^2 + (-3)^2}} = \frac{52}{\sqrt{13}}$

Then the distance from the line to the circle is distance to the center minus the radius of the circle.

$$d = \frac{52}{\sqrt{13}} - \sqrt{13} = \frac{52\sqrt{13}}{13} - \sqrt{13} = 4\sqrt{13} - \sqrt{13} = \mathbf{3\sqrt{13}}$$

Solution 32



Let R be the radius of a circle and s the side of the inscribed square. Then $\frac{S^2}{4} + \frac{S^2}{4} = R^2 \implies S = \sqrt{2}$ R. If s is the side of a square and r the radius of the inscribed circle, then r = $\frac{S}{2}$. Let R_n and S_n the radii and side length of the indicated circles and squares.

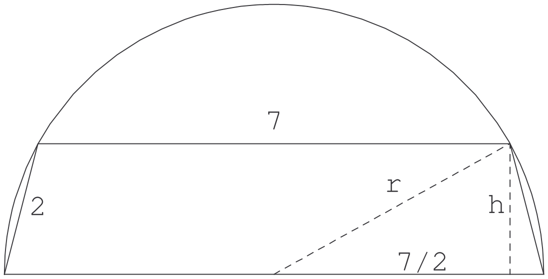
$$R_1 = 1\,,\quad S_1 = \sqrt{2}\,,\; R_2 = \frac{\sqrt{2}}{2}\;,\quad S_2 = \frac{(\sqrt{2})^2}{2},\quad R_3 = \frac{(\sqrt{2})^2}{2^2},\ldots$$

$$S_n = \frac{(\sqrt{2})^n}{2^{n-1}} = \frac{1}{2^{n/2-1}} \quad \text{and} \quad R_n = \frac{(\sqrt{2})^{n-1}}{2^{n-1}} = \frac{1}{2^{n/2-1/2}}$$

If A_n is the area between circle n and square n, then $A_n = \pi R_n{}^2 - S_n{}^2 = \frac{\pi}{2^{n-1}} - \frac{1}{2^{n-2}}$

$$\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{\pi}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{2^{n-2}} = \frac{\pi}{1-\frac{1}{2}} - \frac{2}{1-\frac{1}{2}} = \mathbf{2\pi - 4}$$

Solution 33



From the two indicated right triangles,

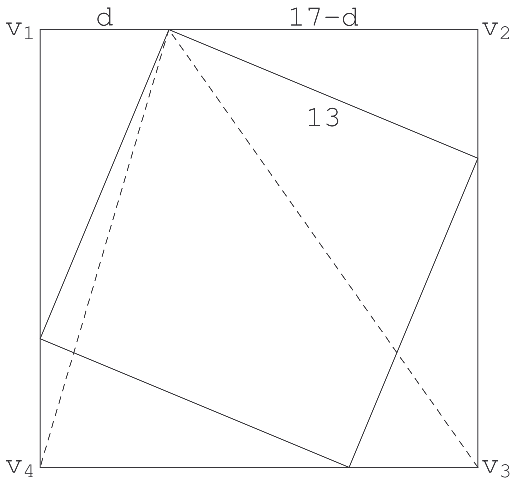
$$\begin{cases} r^2 = h^2 + \left(\frac{7}{2}\right)^2 & (1) \\ 4 = h^2 + \left(r - \frac{7}{2}\right)^2 & (2) \end{cases} \qquad \text{Solving for } h^2$$

$$\begin{cases} h^2 = r^2 - \frac{49}{4} & (1) \\ h^2 = 4 - \left(r - \frac{7}{2}\right)^2 & (2) \end{cases}$$ Equating

$r^2 - \frac{49}{4} = 4 - r^2 + 7\text{r} - \frac{49}{4} \implies 2\,r^2 - 7\text{r} - 4 = 0$

$(2\text{r} + 1)(\text{r} - 4) = 0 \implies \text{r} = 4 \implies$ **d = 8**

Solution 34



From the figure $d^2 + (17 - d)^2 = 13^2 \implies d^2 - 17d + 60 = 0 \implies (d - 12)(d - 5) = 0 \implies d = 5$ and $17 - d = 12$

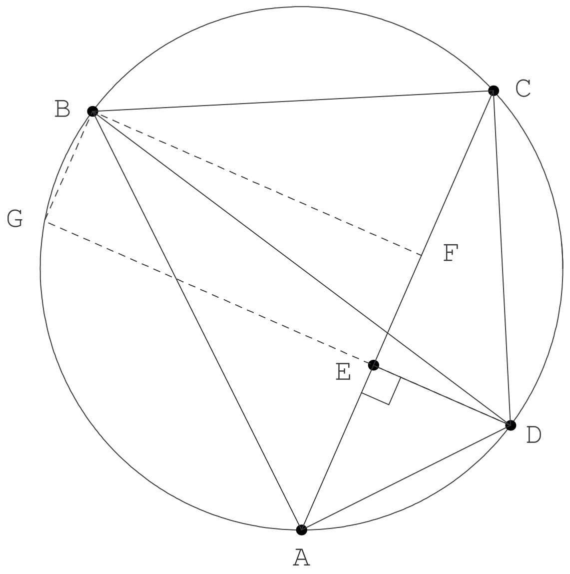
The squares of the distances between the vertices are,

$d\text{ - }v_1 = 25$, $d\text{ - }v_2 = 144$ $d\text{ - }v_3 = 12^2 + 17^2 = 433$ $d\text{ - }v_4 = 5^2 + 17^2 = 314$

So the maximum distance squared is **433** .

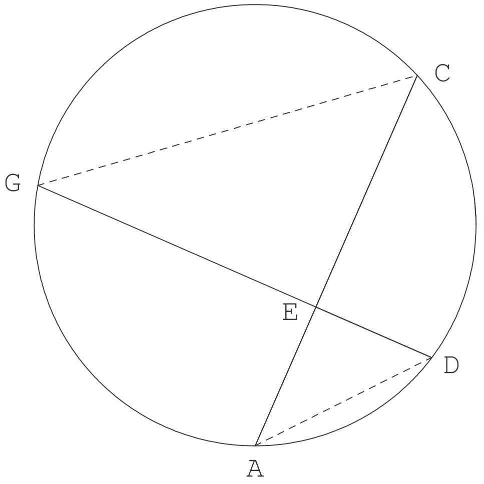
Solution 35

In the given figure, extend DE to the circle at G and drop a perpendicular from B to AC at F.



Since BD is a diameter of the circle, the angle at G is 90°. Since the angles at E and F are 90° by construction, the quadrilateral GBFE is a rectangle. Hence BF = GE.
Since lines AC and GD intersect at E, AE · EC = GE · ED (See below).

Using the given information, $6 \cdot 10 = GE \cdot 5 \implies GE =$ **12 = BF**.



The angles at G and A subtend the same arc so are equal. The vertical angles at E are equal, hence the triangles GCE and AED are similar. Thus the ratios of corresponding sides are equal. $\frac{ED}{EC} = \frac{AE}{GE} \implies ED \cdot GE = AE \cdot EC$.

Solution 36

Consider the powers of 3 and 7 mod 10.

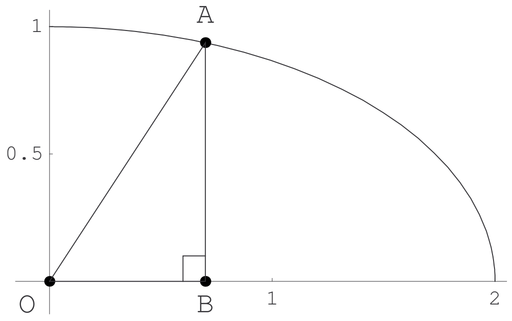
$3^1 \bmod 10 = 3 \quad 7^1 \bmod 10 = 7$
$3^2 \bmod 10 = 9 \quad 7^2 \bmod 10 = 9$
$3^3 \bmod 10 = 7 \quad 7^3 \bmod 10 = 3$
$3^4 \bmod 10 = 1 \quad 7^4 \bmod 10 = 1$

Thus there are 16 pairs of value of 3^a and 7^b mod 10. They are (3,7), (3,9), (3,3), (3,1), (9,7), (9,9), (9,3), (9,1), (7,7), (7,9), (7,3), (7,1), (1,7), (1,9), (1,3), (1,1)

Of these only 3 have a sum which is 6 mod 10, namely (3,3), (9,7) and (7,9).

Thus the probability is $\frac{3}{16}$.

Solution 37



The coordinates of point A are $\left(x, \sqrt{1-\frac{x^2}{4}}\right)$ so the area of the triangle is $\frac{1}{2}\cdot \underline{x} \cdot \sqrt{1-\frac{x^2}{4}} = \frac{1}{2}\sqrt{x^2-\frac{x^4}{4}} = \frac{1}{2}\sqrt{\frac{4x^2-x^4}{4}}$

This will be a maximum when $4x^2-x^4$ is a maximum. Let $t = x^2$. Find maximum value of $4t-t^2 = -(t^2-4t)$. Complete the square,

$-(t^2-4t+4)+4 = -(t-2)^2+4$. Thus the maximum occurs when $t = 2 \implies x = \sqrt{2}$

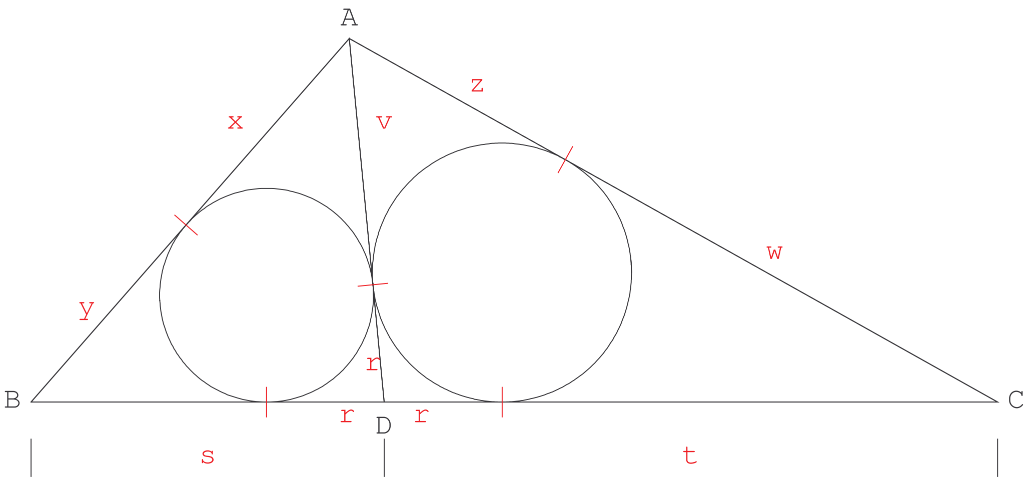
Solution 38

$$\sin(x)+\cos(x) = \frac{1}{2} \implies \sin^2(x)+2\sin(x)\cos(x)+\cos^2(x) = \frac{1}{4} \implies 2\sin(x)\cos(x) = -\frac{3}{4} \implies \sin(x)\cos(x) = -\frac{3}{8}$$

$$(\sin^2(x)+\cos^2(x))^2 = 1 \implies \sin^4(x)+2\sin^2(x)\cos^2(x)+\cos^4(x) = 1$$

$$\sin^4(x)+\cos^4(x) = 1-2\sin^2(x)\cos^2(x) = 1-2\left(\frac{-3}{8}\right)^2 = 1-\frac{9}{32} = \frac{23}{32}$$

Solution 39



Using the fact that tangents to a circle from an external point are equal,

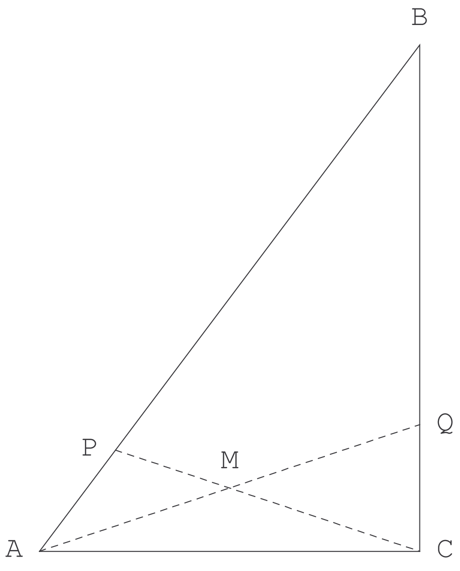
$x = v = z$, $s - r = y$ and $t - r = w$

$$\left\{\begin{array}{l} x + y = AB \\ z + w = AC \\ s + t = AC \end{array}\right. \qquad \text{Using the above}$$

$$\begin{array}{ll} x + s - r = 130 & (1) \\ x + t - r = 200 & (2) \\ s + t = 260 & (3) \end{array} \qquad \text{Combine } (1) - (2) + (3)$$

$2s = 190 \implies$ **$s = 95$**

Solution 40



Let triangle ABC be a 3, 4, 5 right triangle with A = (0,0), B = (3,4) and C = (3,0). The $P = (\frac{3}{5}, \frac{4}{5})$ and Q = (3,1).

Line containing A and Q has equation $y = \frac{1}{3}x$ and line containing P and C has equation $y = \frac{\frac{4}{5}}{\frac{3}{5}-3}(x-3) = -\frac{1}{3}(x-3)$

Equating $\frac{1}{3}x = -\frac{1}{3}(x-3) \Longrightarrow x = \frac{3}{2} \Longrightarrow y = \frac{1}{2}$. $M = (\frac{3}{2}, \frac{1}{2})$.

$$\frac{MC}{PC} = \frac{\sqrt{(\frac{3}{2}-3)^2 + (\frac{1}{2})^2}}{\sqrt{(\frac{3}{5}-3)^2 + (\frac{4}{5})^2}} = \frac{\sqrt{\frac{9}{4}+\frac{1}{4}}}{\sqrt{\frac{144}{25}+\frac{16}{25}}} = \frac{\sqrt{\frac{10}{4}}}{\sqrt{\frac{160}{25}}} = \frac{\frac{\sqrt{10}}{2}}{\frac{4}{5}\sqrt{10}} = \mathbf{\frac{5}{8}}$$

Solution 41

From trigonometric identities for sine and cosine of sum and differences,

$$\left\{\begin{array}{l} \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\ \sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\ \text{--} \\ \sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta) \end{array}\right.$$

$$\left\{\begin{array}{l} \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \sin(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ \text{--} \\ \cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos(\alpha)\cos(\beta) \end{array}\right.$$

$$t = \frac{\sin(\frac{\pi}{9}) + \sin(\frac{2\pi}{9}) + \sin(\frac{4\pi}{9}) + \sin(\frac{5\pi}{9})}{\cos(\frac{\pi}{9}) + \cos(\frac{2\pi}{9}) + \cos(\frac{4\pi}{9}) + \cos(\frac{5\pi}{9})} = \frac{\sin(x) + \sin(2x) + \sin(4x) + \sin(5x)}{\cos(x) + \cos(2x) + \cos(4x) + \cos(5x)} = \frac{[\sin(x) + \sin(5x)] + [\sin(2x) + \sin(4x)]}{[\cos(x) + \cos(5x)] + [\cos(2x) + \cos(4x)]}$$

Letting $\alpha + \beta = 5x$ and $\alpha - \beta = x \Longrightarrow \alpha = 3x$ and $\beta = 2x$

Letting $\alpha + \beta = 4x$ and $\alpha - \beta = 2x \Longrightarrow \alpha = 3x$ and $\beta = x$

$$\frac{[\sin(x) + \sin(5x)] + [\sin(2x) + \sin(4x)]}{[\cos(x) + \cos(5x)] + [\cos(2x) + \cos(4x)]} = \frac{2\sin(3x)\cos(2x) + 2\sin(3x)\cos(x)}{2\cos(3x)\cos(2x) + 2\cos(3x)\cos(x)} = \frac{\sin(3x)\,[\cos(2x) + \cos(x)]}{\cos(3x)\,[\cos(2x) + \cos(x)]} = \frac{\sin(3x)}{\cos(3x)} = \tan(3x)$$

Now with $x = \frac{\pi}{9}$, $t = \tan(\frac{\pi}{3}) = \sqrt{3} \Longrightarrow \mathbf{t^2 = 3}$