UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
FORTY-EIGHTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 3, 2005

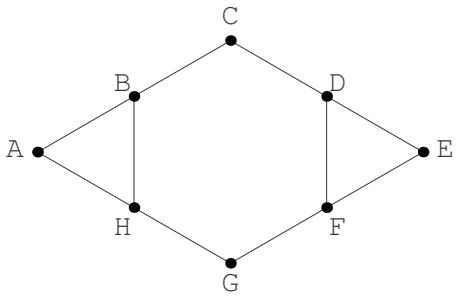
1) Express $\dfrac{\sqrt{\frac{9}{11}}}{\sqrt{5+\frac{9}{11}}}$ as a rational number in lowest terms.

2) Express $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$ as a rational number in lowest terms.

3) A vat is filled with 100 pounds of grapes, which are 99% juice. After workers stomp on the grapes and juice is siphoned off, the grapes are 98% juice. How many pounds of juice were removed?

4) If $A_1 = 3$ and $A_{n+1} = -1 + 2 \cdot A_n$ for $n \geq 1$, what is A_7 ?

5) In the figure, ACEG is a parallelogram and BCDFGH is a regular hexagon. If CG = 10 feet, find AH.



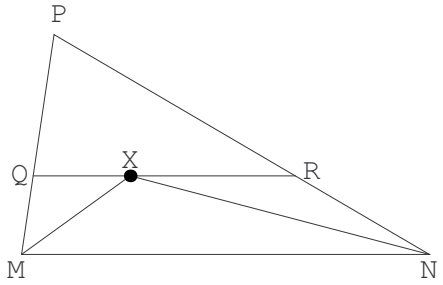
6) Express $\dfrac{100! - 99! - 98!}{100! + 99! + 98!}$ as a rational number in lowest terms.

7) At the grocery store, Larry buys 2 apples, 3 grapefruit and 5 oranges. Larry pays \$9. Karla buys 1 apple, 2 grapefruit and 1 orange. Karla pays \$4. Jack buys 5 apples, 4 grapefruit and 3 oranges. Jack pays \$10. How much does one grapefruit cost? Express your answer in the form \$a.bc where a, b and c are decimal digits.

8) I need 3 working flashlights for a camping trip. In my closet are 4 flashlights that are known to be inoperative and 8 flashlights that are in good working order. If I randomly choose 3 flashlights from the closet, what is the probability that all 3 are in good working order? Express your answer as a rational number in lowest terms.

9) Express $\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}$ as a rational number in lowest terms.

10) What is the perimeter of ΔPQR if PM = 10 cm, MN = 15 cm, PN = 17 cm , QM = QX and XR = RN ?



11) Express $\frac{1}{2} + \frac{1}{\frac{1}{3} + \frac{1}{\frac{1}{4} + 5}}$ as a rational number in lowest terms.

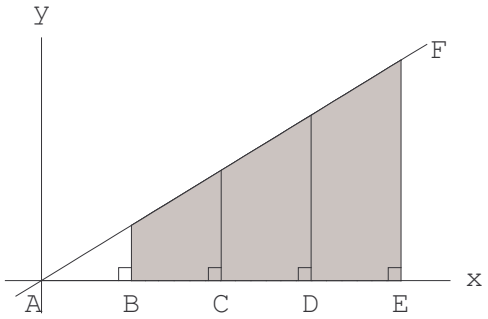
12) Let E be the sum of the even numbers 2, 4, 6, · · ·, 1000. Let O be the sum of the odd numbers 1, 3, 5, · · ·, 999. Find E – O.

13) The degree measures of the four angles of a convex quadrilateral form an arithmetic progression. If the smallest angle has degree measure 75°, what is the degree measure of the largest angle?

14) Suppose that 4 Leghorn hens and 3 New Hampshire Red hens lay as many eggs in 15 days as 3 Leghorn hens and 5 New Hampshire Red hens lay in 12 days. How many days would it take for a Leghorn to produce what a New Hampshire Red can produce in 20 days?

15) A sphere of radius 1unit is inscribed in a right circular cylinder which is in turn inscribed in a sphere. Find the surface area of the outer sphere.

16) Suppose that AB = BC = CD = DE = 7 ft and the area of the shaded region is 18 ft^2. What is the slope of the line segment $\overline{AF}$?



17) How many solutions does the equation $\sin(2x) = 2 \cdot \sin^2(x)$ have in the interval $[\,0\,,\,2\pi\,]$?

18) If 1010101_2 is multiplied by itself, how many of the digits in the resulting base 2 number are ones ?

19) If the letters of BANANA are randomly rearranged, what is the probability that at least one of the six positions will be occupied by the same letter of the alphabet as in the original arrangement? Express your answer as a rational number in lowest terms.

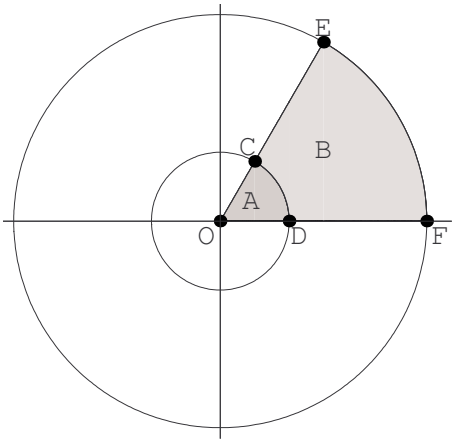
20) Let $y = 6x^2 + 9x + 1$. For which values of x will the y-coordinate be 4 more that twice the x-coordinate?

21) Solve the equation $\frac{3}{x^2 + 7x + 12} + \frac{2}{x^2 - 9} = \frac{4}{x^2 + x - 12}$ for x.

22) Consider two circles of different radii. The difference in the circumferences of the two circles equals the radius of the smaller circle and the difference in the areas of the two circles is $\frac{1}{2\pi} + 2$ square units. What is the radius of the smaller circle?

23) Last year, Ed drove 20,000 miles and spent $1200 on gasoline. This year, Ed intends to reduce his driving mileage by 20% and he estimates that the price per gallon of gasoline has increased by 40% over last year's price. How many dollars should Ed plan to spend on gasoline during the current year? (Assume that the car's efficiency remains constant.)

24) Given two concentric circles, the radius of the larger circle is 60 units. If A is the sector OCD, B is the region CEFD and the area of B is 8 times the area of A, what is the radius of the smaller circle?



25) Suppose that x and y are positive integers such that $xy + 5x + y = 2000$. Find y.

26) Find the area of the region bounded by $y = |x - 1|$ and $y = 2 - |2x - 2|$.

27) Define the operation $\oplus$ by $a \oplus b = \frac{a-b}{a+b}$. Find all values of c such that $(1 \oplus 2) \oplus c = 1 \oplus (2 \oplus c)$.

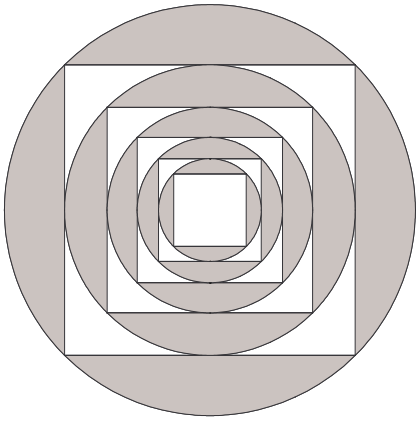
28) Find the smallest positive integer k such that 9! times k is a perfect square .

29) If the sum of the first n terms of a sequence is $n^2 + 5n$, what is the 2005^{th} term of the sequence ?

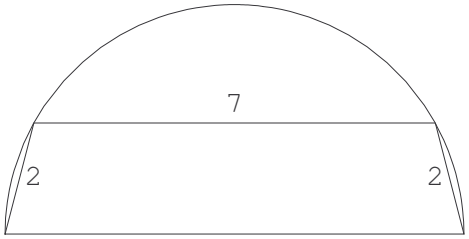
30) Let r_1, r_2 and r_3 be the solutions of $x^3 + 2x^2 - 6x + 9 = 0$. Find $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.

31) Find the shortest distance between the line $y = \frac{2}{3}x + 11$ and the circle $x^2 + y^2 - 4x + 10y + 16 = 0$.

32) Suppose that C_1 is a circle of radius 1, square S_1 is inscribed in C_1, circle C_2 is inscribed in S_1 and square S_2 is inscribed in C_2. This process continues indefinitely, alternating circles and squares. The first 5 circles and squares are shown in the figure. For $n \geq 1$, let A_n be the area of the region inside circle C_n and outside square S_n.
Find $\sum_{n=1}^{\infty} A_n$.

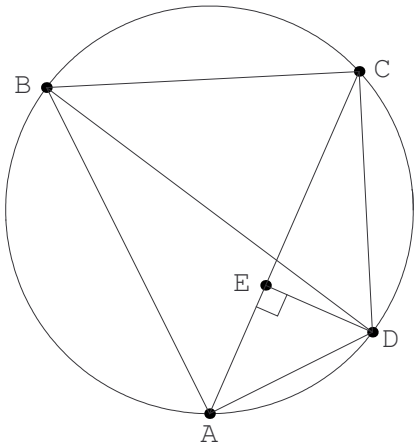


33) As shown in the sketch, an isosceles trapezoid is inscribed in a semicircle. If each of the equal sides has length 2 cm and the shorter of the two parallel sides has length 7 cm, determine the length of the longer parallel side. The longer parallel side is a diameter of the semicircle.



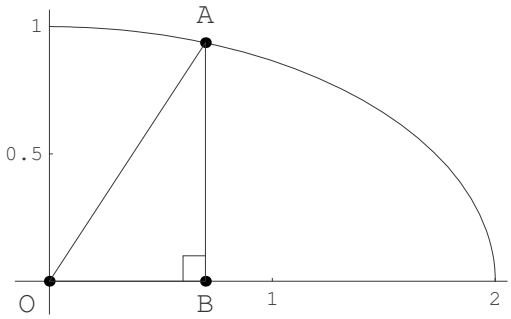
34) A square of perimeter 52 cm is inscribed in a square of perimeter 68 cm. If d is the greatest distance between a vertex of the inner square and a vertex of the outer square, find d^2.

35) Quadrilateral ABCD can be inscribed in a circle in such a way that $\overline{BD}$ is a diameter of the circle. Let E be the point on $\overline{AC}$ such that $\overline{AC}$ and $\overline{DE}$ are perpendicular. If AE = 6, EC = 10 and DE = 5, find the perpendicular distance from B to chord $\overline{AC}$.



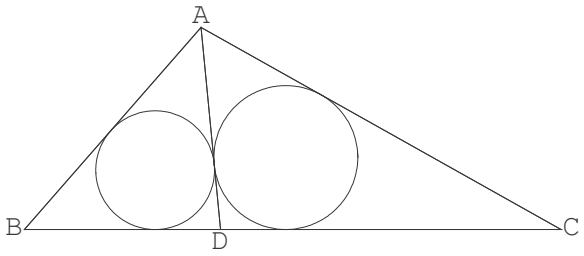
36) Suppose that a and b are two integers, not necessarily distinct, chosen from the set S = { 1, 2, 3, · · · , 200 }. What is the probability that the units digit of $3^a + 7^b$ is 6 ? Express your answer as a rational number in lowest terms.

37) Point A is chosen on the portion of the ellipse $\frac{x^2}{4} + y^2 = 1$ that lies in the first quadrant. Line $\overline{AB}$ is drawn perpendicular to the x-axis at point B. Find the length of $\overline{OB}$ such that the area of triangle OAB is as large as possible.



38) If $\sin(x) + \cos(x) = \frac{1}{2}$, express $\sin^4(x) + \cos^4(x)$ as a rational number in lowest terms.

39) In triangle ABC, AB = 130, AC = 200 and BC = 260. Point D is chosen on BC so that the circles inscribed in triangles ABD and ADC are tangent to $\overline{AD}$ at the same point. Find BD.



40) In ΔABC, a point P is chosen on side $\overline{AB}$ so that AP : PB = 1 : 4 and a point Q is chosen on side $\overline{BC}$ so that CQ : QB = 1 : 3. Segments $\overline{CP}$ and $\overline{AQ}$ intersect at M. Express the value of $\frac{MC}{PC}$ as a rational number in lowest terms.

41) If $t = \frac{\sin(\frac{\pi}{9}) + \sin(\frac{2\pi}{9}) + \sin(\frac{4\pi}{9}) + \sin(\frac{5\pi}{9})}{\cos(\frac{\pi}{9}) + \cos(\frac{2\pi}{9}) + \cos(\frac{4\pi}{9}) + \cos(\frac{5\pi}{9})}$, express t^2 as a rational number in lowest terms.