Universality of the Mandelbrot set and Julia sets

The previous discussion on the Mandelbrot set and Julia sets is for the quadratic map \( f_c(z) = z^2 + c \) only. A surprising fact is that such sets are found in many other one-parameter families of analytic mappings as well. In other words, these sets are "universal". The reason is that the mappings locally may behave as a quadratic polynomial. This is explained by Douady and Hubbard (1985) in their paper about polynomial-like mappings.

Example 1. \( g_c(z) = z^4 + c \).

This map also has one critical point \( z = 0 \).

Similarly define \( E = \{ c : g_c^n(0) \to \infty \} \).

which is the counterpart of the Mandelbrot set in quadratic maps.

This set is shown below. It looks like three Mandelbrot sets merged together.

![The set E.](image)

Note that the orbit \( \{ g_c^n(0) \} \) is

\[
0 \to c \to c^4 + c \to (c^4 + c)^4 + c \to \ldots \ldots
\]

Under the rotation \( c \to ce^{2\pi i/3} \),

60
this orbit is also rotated by $120^\circ$. Thus the set $E$ is invariant under $120^\circ$ rotation. This is why $E$ has three "heads" and three "bodies". The boundary of the central merged three cardioids is given by

\[
\begin{cases}
z^4 + c = z \\
|4z^3| = 1.
\end{cases}
\]

which is \[c = \frac{1}{4} e^{i\theta} - \frac{1}{4} e^{4i\theta}, \quad 0 \leq \theta < 2\pi.\]

It is shown below.

![Boundary of the central region in set E](image)

When $c$ is inside this region, the Julia set would be a quasi-circle. When $c$ is in the other parts of the set $E$, the Julia set is quite similar to that of the quadratic map $(z^2 + c)$ for $c$ at similar location of the Mandelbrot set. Some examples are shown on the next page.

Consider the polynomial equation

\[ p(z) = z(z - 1)(z - \rho) = 0 \]

where \( \rho \) is a complex parameter. This equation has three roots, 0, 1, \( \rho \).

When Newton's method is applied to it, we get

\[ z_{n+1} = N(z_n), \]

where \( N(z) = z - \frac{p(z)}{p'(z)} \).

Notice that

\[ N'(z) = \frac{p(z)p''(z)}{p'(z)^2} = \frac{6z(z-1)(z-\rho)(z-\rho_{-1})}{p'(z)^2}, \]

thus map \( N \) has four critical points

0, 1, \( \rho \) and \( \rho_{-1} \).

The first three critical points are also the super-attracting fixed points of map \( N \). But the last one, \( \rho_{-1} \), is free.

An interesting fact about map \( N \) is that, for certain parameter values \( \rho \), \( N(z) \) has an attracting cycle other than the super-attracting fixed points 0, 1 and \( \rho \). In such cases, every starting point in the basin of attraction of that attracting cycle leads to an unsuccessful application of Newton's method.

1. The parameter set of \( \rho \) where the attracting cycle exists.

First we determine for what \( \rho \) values such an attracting cycle exists. Recall that for any rational map, every attracting cycle attracts at least one critical point (Fatou). The critical points 0, 1 and \( \rho \) of map \( N \) are super-attracting fixed points. Thus they (as starting points) will not lead to new attracting cycles. But the free critical point \( \frac{1+\rho}{3} \) may.

Thus, our strategy is to examine the orbit of the free critical point \( \frac{\rho_{-1}}{3} \).

For this purpose, we define the set

\[ B_\rho = \{ \rho : N^n(\frac{\rho_{-1}}{3}) \to \text{any one of 0, 1 and } \rho \}. \]
This set is shown as the black region in the figure below, followed by two successive enlargements.

![Figure 1: The set of $B_\rho$ (shown as black)](image)

![Figure 2](image)

![Figure 3](image)

It is a Mandelbrot bug right in front of our eyes!

When $\rho$ is in the black cardioid (Fig 3), map $N$ has an attracting two-cycle.

In the largest bud attached to it, $N$ has an attracting four-cycle, and so on.

Next we discuss the basins of attraction for the existing attractors. For any fixed $\rho$, define
\[ K_\rho = \{ z : \quad N_\rho^n(z) \to \text{any one of the roots } 0, 1 \text{ and } \rho \}. \]

It is easy to see that if \( \rho \) is in the interior of \( B_\rho \), then \( K_\rho \) also has non-empty interior. When \( \rho = 0.909419 + 0.416106i \), this set \( K_\rho \) is shown in Fig. 4 as the black region. An enlargement of a small rectangle in Fig. 4 is shown in Fig. 5. (Note: This \( \rho \) is in the largest bud in Fig. 3. At this value, \( N \) has a superattracting four-cycle.)

![Fig. 4: The set \( K_\rho \) for \( \rho = 0.909419 + 0.416106i \). It is marked as black. The other points are shaded corresponding to the number of iterates necessary for its orbit to converge to one of the roots (up to a reasonable accuracy). The box specifications are: Lower left corner: \(-0.790167 - 0.907198i\) Upper right corner: \(1.83213 + 1.55605i\)](image1)

![Fig. 5: Enlargement of a rectangle containing one of the black regions in Fig. 4. The box dimensions are: Lower left corner: \(0.471402 - 0.023518\) Upper right corner: \(0.807454 + 0.318906i\)](image2)