Properties of Julia sets

In this section, we discuss some general dynamical properties of Julia sets for polynomial maps. Of course, these properties apply to the quadratic map $f_c$ as a special case.

Suppose $P(z)$ is a polynomial function, then its Julia set $J(P)$ has the following properties:

1. Let $P$ be a polynomial of degree $n \geq 2$. Then $J(P) \neq \emptyset$, i.e. nonempty.
2. $J(P) \subseteq K(P)$, where $K(P)$ is the filled-in Julia set of $P$.
3. $J(P) = J(P^n)$, where $n$ is a positive integer.
4. $J(P)$ is completely invariant, i.e. if $z \in J(P)$, then $P(z)$ and $P^{-1}(z)$ are also in $J(P)$.
5. Let $z_0 \in J(P)$, then

$$J(P) = \text{closure} \left( \bigcup_{k=0}^{\infty} P^{-k}(z_0) \right),$$

i.e. $J(P)$ is the closure of all the preimages of $z_0$.

6. $J(P)$ is the closure of the set of repelling periodic points of $P$.
7. $J(P)$ has empty interior.
8. $P$ is chaotic on $J(P)$.

Note: Property 5 is often used to plot the Julia sets graphically, and property 6 is often used as an alternative definition for the Julia set.

Most of the above properties on Julia sets can be extended to rational maps. Refer to Devaney (1986) and Devaney (1994) for more details.

Geometry of Julia sets.

In this section, we return to the quadratic map

$$f_c(z) = z^2 + c$$

and discuss the geometry of typical Julia sets for some chosen $c$ values. A classification on the geometry of Julia sets in the $c$-plane will be covered in the next section.

First of all, Julia sets $J_c$ have the following geometric symmetries:

1. For any value of $c$, $J_c$ is symmetric about the origin, i.e.

   if \quad $z_0 \in J_c$, so does $-z_0$.

   Proof: $z_0 \in J_c \Rightarrow z_0^2 + c \in J_c$. But \quad $f_c^{-1}(z_0^2 + c) = \{z_0, -z_0\} \Rightarrow -z_0 \in J_c$.

   (this proof is based on the invariance property of Julia sets.)
2. If \(c\) is real, then \(J_c\) is symmetric about the real \((z)\) axis, i.e.

\[
\text{if } z_0 \in J_c, \text{ so does } z_0^*.
\]

Proof: If \(z \in k_c \Rightarrow \{f_c^n(z)\}\) is bounded \(\Rightarrow \{f_c^n(z^*)\}\) is bounded
\[
\Rightarrow \{f_c^n(z^*)\}\text{ is bounded } \Rightarrow z^* \in k_c.
\]

Thus \(k_c\) is symmetric about the real \((z)\) axis.

It follows that \(J_c = \partial k_c\) is symmetric about real \((z)\) axis.

3. If \(c\) is real, then \(J_c\) is also symmetric about \(\text{Im}(z)\) axis, i.e.

\[
\text{if } z_0 \in J_c, \text{ so does } -z_0^*.
\]

Proof: If \(z_0 \in J_c\), according to the first and second symmetries,
\[
-z_0 \in J_c \Rightarrow -z_0^* \in J_c.
\]

The geometry of Julia sets depends crucially on the attracting orbit of the map. This will be illustrated in the following examples.

Example 1. \(f_0(z) = z^2\).

In this case, \(f_0\) has two fixed points, 0 and 1.

0 is attracting and 1 repelling.

The filled-in Julia set is clearly \(k_0 = \text{unit disk}\)

and \(J_0 = \partial k_0 = \text{unit circle, a smooth closed curve}\).

Suppose \(c \neq 0\) but small \((|c| < \frac{1}{4})\). In this case, the two fixed points of \(f_c\) are

\[
\frac{1 - \sqrt{1 - 4c}}{2} \text{ and } \frac{1 + \sqrt{1 - 4c}}{2}.
\]

It is easy to see that the first one is attracting and the second one repelling. The filled-in Julia set \(K_c\) is also one piece, and \(J_c\) a closed curve. But \(J_c\) now contains no smooth arcs. In fact, it is a fractal. Its box and Hausdorff dimensions are (Falconer 1990):

\[
\text{dim}_{\text{B}} J_c = \text{dim}_{\text{H}} J_c = 1 + \frac{\log 2}{\log 2} + (|c|^2).
\]

Hence, the larger \(c\) is, the more irregular \(J_c\) is.

Example 2. \(f_1(z) = z^2 - 1\).
In this case, \( f_{-1} \) has two repelling fixed points \( \frac{1 \pm i \sqrt{3}}{2} \) and an attracting two-cycle \( \{0, -1\} \). (It is super attracting actually.) How does its Julia set look like? It is shown in Fig 6.4 below. This set is also a fractal, but it is no longer a single curve. Rather it has infinite connected components. In addition, at each joint, two arms are connected.

![Diagram of the Julia set](image)

**Fig. 6.4.** The Julia set of \( P(z) = z^2 - 1 \).

One way to understand this peculiar Julia set is to trace the preimages of the unstable fixed point \( z_0 = \frac{1 - i \sqrt{3}}{2} = -0.618... \). Solving the equation

\[
z^2_{(n+1)} - 1 = z_n
\]

recursively, it is easy to find out that

\[
z_{-1} = \pm z_0 = \pm 0.618...
\]
\[
z_{-2} = \pm 1.272..., \pm 0.618...
\]
\[
z_{-3} = \pm 1.5073..., \pm 0.5216...i, \pm 1.272..., \pm 0.618...
\]

......

We can see that the preimages of \( z_0 \) gradually spread out in the Julia set. If we only trace the preimages along the positive real \( (x) \) axis, we get

\[
z_0 = 0.6180...
\]
\[
z_{-1} = 0.6180...
\]
\[
z_{-2} = 1.2720...
\]
\[
z_{-3} = 1.5073...
\]
\[
z_{-4} = 1.5835...
\]
\[
z_{-5} = 1.6073...
\]
\[
z_{-6} = 1.6147...
\]
\[
z_{-7} = 1.6170...
\]
\[
z_{-8} = 1.6177...
\]
\[
z_{-9} = 1.6179...
\]

......
This sequence \( \{z_n\} \) approaches a limit as \( n \to \infty \). The limit \( z_\infty \) can be obtained from the equation

\[
z_\infty^2 - 1 = z_\infty,
\]

which gives \( z_\infty = \frac{1 + \sqrt{5}}{2} = 1.618... \). This is just the other fixed point of \( f_{-1}(z) \). Now it is clear that \( \frac{1 + \sqrt{5}}{2} \) is the right gate of the Julia set \( J_{-1} \). By symmetry and preimage argument,

\[
-\frac{1 + \sqrt{5}}{2} = -1.618... \text{ is the left gate of } J_{-1}.
\]

\[
\sqrt{\frac{\sqrt{5} - 1}{2}} i = 0.786... i \text{ is the top gate and }
\]

\[
-\sqrt{\frac{\sqrt{5} - 1}{2}} i = -0.786... i \text{ the bottom gate.}
\]

Other special joint points in \( J_{-1} \) are all preimages of \( z_0 = \frac{1 - \sqrt{5}}{2} \) and can be traced respectively.

Another way to comprehend the geometric structures of this Julia set and filled-in Julia set is by tracing the preimages of the interval \([0, \frac{1 + \sqrt{5}}{2}]\). Since this interval falls entirely in the filled-in Julia set \( J_{-1} \), so must its preimages. In addition, since \( \frac{1 + \sqrt{5}}{2} \) is both a boundary point of \( k_{-1} \) and an accumulation point of \( z_0 \)’s preimages, such local structure of \( k_{-1} \) around \( \frac{1 + \sqrt{5}}{2} \) will be carried over to all the preimages of this interval. These preimages up to the fifth level are shown in Fig 1. It is important to realize that these preimage curves reveal the orientation and overall structure of the filled-in Julia set and Julia set itself.

Example 3. \( f_c(z) = z^2 + c \) with \( c = 0.12 + 0.74i \).

In this case, \( f_c \) has an attracting 3-cycle. Its Julia set is shown in Fig 13.37. Observe that at every point, three pieces are connected. The self-similarity in the Julia set is also revealed in the figure by image tracing.

![Figure 1: Preimages of the interval I: [0,(1+√5)/2]
in the quadratic map f(z) = z^2 - 1. Red: I; pink:
f^{-1}(I); cyan: f^{-2}(I); blue: f^{-3}(I); green: f^{-4}(I);
black: f^{-5}(I).](image)
Self-Similarity of a Julia Set

The self-similarity of the Julia sets. These two pictures show how a very small section of the Julia set, denoted by \( R_{-7} \), is transformed several times. In each transformation the covered portion of the Julia set indicated by the bold black parts labeled \( R_{-6} \) to \( R_{-1} \), increases. After six iterations the result \( R_{-1} \) is already one half of the Julia set; one more application of \( z \rightarrow z^2 + c \) yields the whole set \( R_0 \).

\[
C = 0.12 + 0.74i
\]

In this picture.

Figure 13.37
Properties of the Mandelbrot Set

(For coverage of this part at a higher level, refer to Devaney (1994)).

The Mandelbrot set is

\[ M = \{ c : f_c^2(0) \to \infty \} \text{, where } f_c(x) = x^2 + c, \]

and is shown below.

The Mandelbrot Set — Old and New Rendering

The insert shows an original printout from Mandelbrot's experiment. We have produced the large Mandelbrot set using a modern laser printer and a more accurate mathematical algorithm.

![Mandelbrot Set Image]

Figure 14.3

This set is very complicated. In this section, we discuss some of its properties.

1. \( M \) is a connected set. (Douady and Hubbard 1982)
2. \( J_c \) is connected \( \iff c \in M \). \( J_c \) is a topological cantor set \( \iff c \notin M \). (Julia, Fatou)
3. \( M \subset \) the disk of radius 2, i.e. \( \{ c : |c| \leq 2 \} \).

Proof: If \( |c| > 2 \), then \( f_c(0) = c, \ f_c^2(0) = c^2 + c \)

\[ \Rightarrow |f_c^2(0)| \geq |c|^2 - |c| = |c|(|c| - 1) > |c| > 2 \]

\[ f_c^2(0) = f_c^2(0)^2 + c \]

\[ \Rightarrow |f_c^2(0)| \geq |f_c^2(0)|^2 - c = |f_c^2(0)|(|f_c^2(0)| - \frac{c}{|f_c^2(0)|}) > |f_c^2(0)|(|c| - 1) \]

\[ \geq |c|(|c| - 1)^2. \]

.........
In general, we can show that

\[ |f_c^2(0)| > |c|(|c| - 1)^{n-1}. \]

Thus, as \( n \to \infty \), \( |f_c^n(0)| \to \infty \Rightarrow c \notin M. \)

By the way, we can similarly show that

\[ J_c \subset \{ z : |z| \leq \max(2, |c|) \}. \]

Proof: If \( |z| > \max(2, |c|) \),

then \( |f_c(z)| = |z^2 + c| \geq |z|^2 - |c| = |z|(|z| - \frac{|c|}{|z|}) > |z|. \)

\[ |f_c(z)| \geq |f_c(z)|(|f_c(z)| - \frac{|c|}{|f_c(z)|}) \]

\[ > |f_c(z)|(|z| - \frac{|c|}{|f_c(z)|}) \]

\[ \geq |z|(|z| - \frac{|c|}{|z|})^2 \]

\[ \ldots \]

\[ |f_c^n(z)| > |z|(|z| - \frac{|c|}{|z|})^n. \]

Thus, as \( n \to \infty \), \( f_c^n(z) \to \infty \Rightarrow z \notin J_c \Rightarrow z \notin J_c. \)

Next, we discuss the buds in the Mandelbrot set.

Each bud consists of \( c \) values for which \( f_c \) has an attracting cycle of some given period \( k \). Thus, inside of it, we have

\[ f_c^k(z) = z \]

and \( |(f_c^k)'(z)| < 1. \)

Its boundary can be obtained by solving the two equations

\[
\begin{cases}
  f_c^k(z) = z \\
  |(f_c^k)'(z)| = 1
\end{cases}
\]

simultaneously.
1. The main cardioid $C_1$

In this region, $f_c$ has an attracting fixed point. Thus its boundary is given by

\[
\begin{align*}
z^2 + c &= z \\
|2z| &= 1
\end{align*}
\]

It is easy to check that this curve is

\[c = \frac{1}{2} e^{i\theta} - \frac{1}{2} e^{2i\theta}, \quad 0 \leq \theta < 2\pi.\]

2. The period–2 bud $C_2$

In this region, $f_c$ has an attracting two-cycle. Its boundary is given by

\[
\begin{align*}
(z^2 + c)^2 + c &= z \\
|4z(z^2 + c)| &= 1
\end{align*}
\]

The first equation can be factorized as

\[(z^2 + c - z)(z^2 + z + 1 - c) = 0.
\]

By using this equation, the relation

\[|4z(z^2 + c)| = 1\]

can be simplified to be

\[|c + 1| = \frac{1}{4}.
\]

Thus the boundary of bud $C_2$ is a circle of radius $\frac{1}{4}$ centered at $c = -1$. This bud and the main cardioid meet at the point $c = -\frac{1}{4}$. As $c$ passes from $C_1$ to $C_2$ through this point, $f_c$ undergoes a period-doubling bifurcation.

**Question:** In general, at what points do buds and the main cardioid $c_1$ meet?

The answer is simple: they meet at points

\[c = \frac{1}{2} e^{i\theta} - \frac{1}{2} e^{2i\theta}\]

with $\theta = 2\pi p/q$, where $p$ and $q$ are integers.

Furthermore, the bud which meets $C_1$ at $\theta = 2\pi p/q$ is a period–$q$ bud. Sometimes it is called a $p/q$–bud.
Two observations about a $p/q - \text{bud}$:

1. A large antenna grows outward from each $p/q - \text{bud}$. This antenna has exactly $q$ branches. As an example, the antenna for the $2/5 - \text{bud}$ is shown below.

![Antenna for 2/5-bud](image)

2. In a $p/q - \text{bud}$, the Julia sets have $q$ components connected by single points.

For example, when $c = -0.5 + 0.55i$ which is in the $2/5 - \text{bud}$, the Julia set is shown below.

![Julia set for 2/5-bud](image)

**Secondary and higher-degree buds of $M$ and their Julia sets**

Now we discuss secondary and higher-degree buds of $M$ and their Julia sets. It is known that secondary buds attached to the boundary of a primary $p/q - \text{bud}$ have periods which are multiples of $q$. For instance, on the boundary of the primary $\frac{1}{3} - \text{bud}$, the largest secondary bud (denoted as $A$ in Figure 1 below) has period $3 \cdot 2 = 6$. As $c$ moves from the $\frac{1}{3} - \text{bud}$ to bud $A$, $f_c$ undergoes a period-doubling bifurcation. The second largest buds (denoted as $B$ and $C$) have period $3 \cdot 3 = 9$. As $c$ moves from the $\frac{1}{3} - \text{bud}$ to bud $B$ or $C$, $f_c$ undergoes a period-tripling bifurcation. In general, if a bud has period $q$, then the smaller buds attached on its boundary have periods being multiples of $q$. 

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When $c$ is in secondary or higher-degree buds of $M$, the structure of Julia sets will be more intricate. If $c$ is in a secondary bud of period $k \cdot q$ which is attached to a primary $p/q$ – bud, then the Julia set will have two arm patterns within it: $q$ arms connected to a point and $k$ arms connected to a point. For example, the Julia sets for two $c$-values, one in bud $A$ and the other one in bud $B$ (see Fig. 1), are displayed below.

$J_c$: $c \in$ bud $A$

$J_c$: $c \in$ bud $B$

I have not explored Julia sets in higher-degree buds. This will be left as an exercise.
Minature copies of the Mandelbrot set and their Julia sets

Along the "antennas" of the Mandelbrot set, we can find small copies of the entire set $M$. The largest miniature copy is on the negative real ($c$) axis. It is shown below. When $c$ is in its cardioid (denoted as $U$),

![Image](image.png)

Figure 3. The largest miniature copy of the Mandelbrot set on the real axis of $c$. (the counterpart of the largest period-3 window in the real map).

$f_c$ has an attracting 3-cycle, while the Julia set has infinite quasi-circles of various sizes stringed together (see Fig. 4a below). When $c$ is in bud $V$, $f_c$ has an attracting 6-cycle. As $c$ moves from $U$ to $V$, a period-doubling bifurcation occurs. The Julia sets in $V$ have infinite pieces of the central component in Fig. 4b which are all connected. The exploration of other buds in the miniature copy of $M$ (Fig. 3) and other miniature copies of $M$ will be left as an exercise.

![Image](image.png)

**Fig 4a:** A Julia set when $c$ is in the secondary cardioid $U$.

![Image](image.png)

**Fig. 4b:** A Julia set when $c \in$ bud $V$.

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Boundary points of $M$ and their Julia sets

The boundary of $M$ is as intricate as $M$ itself, if not more. For $c$ values on $M$'s boundary, the Julia sets have many different patterns. The classification of those Julia sets is delicate, and we do not intend to do it here. Instead, we will just show some typical Julia sets for certain $c$ values in order to get a general idea.

(1) The boundary of the main cardioid 
\[ c = \frac{1}{2} e^{i\theta} - \frac{1}{4} e^{2i\theta}, \quad 0 \leq \theta < 2\pi. \]

Write $\theta = 2\pi \alpha$. If $\alpha = p/q$ is a rational number, then $c$ is the attachment point of the primary $p/q$-bud to the main cardioid. In this case, the Julia set is similar to that in the $p/q$-bud. For instance,

\[ \alpha = \frac{1}{4} \Rightarrow c = -0.75 \]
\[ \alpha = \frac{1}{7} \Rightarrow c = -0.125 + 0.649519...i \]

Their Julia sets are shown below.

![Julia sets](image)

$J_c$: $c = -0.75$

$J_c$: $c = -0.125 + 0.649519...i$

But if $\alpha$ is irrational, the Julia set will be more subtle.

Example: \[ \alpha = \frac{\sqrt{5} + 1}{2}, \quad c = -0.3905407802... -0.5867879073...i. \]

The Julia set is given below. It is called a Siegel Disk.

A Siegel Disk

The Julia sets that belong to $c \approx -0.3905407802... -0.5867879073i$ is a Siegel disk. With $\alpha$ being the golden mean, use $\phi = 2\pi \alpha \approx 3.883222077$ in eqn. (14.4) on page 858 to obtain the real and imaginary components of $c$. The dynamics near the fixed point is characterized by invariant curves on which the iteration acts like a rotation by the angle $\alpha$.

Figure 14.22

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(2) Misiurewicz points and their Julia sets

The boundary of $M$ contains a class of special infinite points which are called Misiurewicz points.

Definition: For a value of $c$, if $z = 0$ is pre-periodic (but not periodic) under the map $f_c$, then $c$ is called a Misiurewicz point.

Example 1. $c = i$ is a Misiurewicz point

Proof: $f_c^2 = z^2 + i$

$$0 \rightarrow i \rightarrow -1 + i \rightarrow -i \rightarrow -1 + i \rightarrow -i \ldots .$$

Similarly, $c = -2$ is also a Misiurewicz point.

Misiurewicz points are clearly in $M$ because $\{f_c^2(0)\}$ is bounded. In addition, the following results have been proven (Douady and Hubbard 1984):

(i) If $c$ is a Misiurewicz point, then the corresponding periodic cycle is repelling.
(ii) If $c$ is a Misiurewicz point, then $K_c = J_c$, i.e. the filled-in set has no interior.
(iii) Misiurewicz points are dense on the boundary of $M$. In other words, for any point on the boundary of $M$, an arbitrarily small disk around that point contains at least one Misiurewicz point.

The Julia set for $c = i$ is a dendrite (an infinitely branched curve) which is shown below.

![Julia Set for $c = 0.6000 + 1.8000i$](image)
The most important and surprising result about Misiurewicz points is a theorem proved by Tan Lei in 1989. Suppose \( c \) is a Misiurewicz point, and \( z_1 \) is a point on the periodic cycle of period \( p \) in the orbit \( \{ f^k(0) \} \). Define the multiplier \( p \) of \( c \) by

\[
p = (f_{z_1}^p)'(z_1).
\]

Then Tan Lei's result can be stated as follows:

**Theorem:** If \( c \) is a Misiurewicz point, then

1. The Julia set \( J_c \) and the Mandelbrot set are both asymptotically self-similar at the point \( c \) using the same multiplier \( p \).
2. The associated limit objects \( L_J \) and \( L_M \) are essentially the same; they differ only by some scaling and a rotation \( (L_M = \lambda L_J, \text{ where } \lambda \text{ is a suitable complex number}) \).

**Demonstration of Tan Lei's theorem**

Take \( c = i \). The orbit of zero is: \( 0 \rightarrow i \rightarrow -1 + i \rightarrow -i \rightarrow -1 + i \rightarrow -i \)

Thus \( p = 2(-1 + i) \cdot 2(-i) = 4 + 4i = 4\sqrt{2}e^{i\pi} \).

(1) According to Tan Lei's result (first part), asymptotically the Julia set centered at the point \( i \), when amplified by a factor \( 4\sqrt{2} \) and rotated by an angle \( 45^\circ \), should remain the same. This is confirmed in the following figure.

*Closeup at \( c = i \)*

Successive magnification by a scaling factor \( 4\sqrt{2} \) reveal a rotation by 45 degrees from one image to the next.

*Figure 14.43*
Similar features can also be observed in the Mandelbrot set around point $i$.

(2) Tan Lei’s result (second part) says that the microstructures of the Mandelbrot set and Julia set around the point $i$ are essentially the same except some scaling and a rotation. This can be seen from the next figures.