Computing Annihilators of Class Groups from Derivatives of L-functions

Jonathan W. Sands and Brett A. Tangedal

University of Vermont and University of North Carolina at Greensboro

3 October 2015
Brumer’s Conjecture: When $K/F$ is an abelian extension of number fields with Galois group $G$, Brumer’s conjecture states that the values at the origin of the Artin $L$-functions for $K/F$ can be used to provide annihilators in $\mathbb{Z}[G]$ of the ideal class group of $K$. 

Limitation: The annihilators obtained this way are 0 unless $F$ is totally real and $K$ is totally complex.

Key Question: Can one obtain non-trivial annihilators of class groups from the behavior of $L$-functions at the origin for all abelian extensions of number fields?
Motivating Question

**Brumer’s Conjecture:** When $K/F$ is an abelian extension of number fields with Galois group $G$, Brumer’s conjecture states that the values at the origin of the Artin $L$-functions for $K/F$ can be used to provide annihilators in $\mathbb{Z}[G]$ of the ideal class group of $K$.

**Limitation:** The annihilators obtained this way are 0 unless $F$ is totally real and $K$ is totally complex.
Motivating Question

**Brumer’s Conjecture:** When $K/F$ is an abelian extension of number fields with Galois group $G$, Brumer’s conjecture states that the values at the origin of the Artin $L$-functions for $K/F$ can be used to provide annihilators in $\mathbb{Z}[G]$ of the ideal class group of $K$.

**Limitation:** The annihilators obtained this way are 0 unless $F$ is totally real and $K$ is totally complex.

**Key Question:** Can one obtain non-trivial annihilators of class groups from the behavior of $L$-functions at the origin for all abelian extensions of number fields?
- $F = \text{algebraic number field (finite degree over the rationals)}.$
- $K = \text{finite abelian Galois extension of } F$
- $G = \text{Galois group of } K \text{ over } F$
- $S = \text{Finite set of primes of } F \text{ including all infinite primes and all primes ramifying in } K$
- $\mathcal{O}_K^S = \text{"S-integers of } K\text{"}, \text{elements of } K \text{ whose valuation is non-negative at each prime not lying above a prime of } S.$
- $\text{Cl}_K^S = \text{"S-class group of } K\text{"}, \text{nonzero fractional ideals of } \mathcal{O}_K^S \text{ modulo principal nonzero fractional ideals. It is a module over } \mathbb{Z}[G].$
- $\text{Cl}_K = \text{usual class group for } S \text{ containing no finite primes.}$
Artin L-Functions

- $\psi \in \hat{G} =$ group of characters of the abelian group $G = \text{Gal}(K/F)$.
- $\sigma_p \in G$ is the Frobenius automorphism for a prime $p$ of $F$ that is not in $S$ (therefore unramified in $K$).
- $Np =$ absolute norm of the ideal $p$.
- $L_S(s, \psi) = \prod_{\text{prime } p \notin S} (1 - \frac{1}{Np^s} \psi(\sigma_p))^{-1}$ for $\text{Re}(s) > 1$, extends meromorphically to $\mathbb{C}$.
The Stickelberger Element

\[ e_\psi = \frac{1}{|G|} \sum_{\sigma \in G} \psi(\sigma)\sigma^{-1}, \text{ the } \psi\text{-idempotent in } \mathbb{C}[G] \]

\[ \theta^S_{K/F}(0) = \sum_{\psi \in \hat{G}} L_S(0, \psi^{-1}) e_\psi, \text{ the } S\text{-Stickelberger element for } K/F. \]

\[ \theta^S_{K/F}(0) \in \mathbb{Q}[G], \text{ by Klingen-Siegel} \]
Brumer’s Conjecture

- \( \mu_K \) = group of roots of unity of \( K \), a module over \( \mathbb{Z}[G] \).
- \( w_K = |\mu_K| \), the order of this group.
- \( \alpha = \) any annihilator of \( \mu_K \) in \( \mathbb{Z}[G] \), e.g. \( w_K \).
- \( \alpha \theta^S_{K/F}(0) \in \mathbb{Z}[G] \) by Deligne-Ribet and Cassou-Nogues.

Conjecture (Brumer)

\( \alpha \theta^S_{K/F}(0) \) annihilates \( \text{Cl}_K \).

Known to be true for \( F = \mathbb{Q} \) by Stickelberger’s Theorem. Greither-Popescu show that the \( p \)-part holds when \( K \) is CM, \( S \) contains all primes above \( p \neq 2 \), and the Iwasawa invariant \( \mu_p = 0 \).
Unless $F$ is totally real and $K$ is totally complex or $|S| = 1$, the previous conjecture is vacuous because $\theta^S_{K/F}(0) = 0$.

Producing an annihilator involves using all $\chi$ for which $L_S(0, \chi) \neq 0$.

Results of David Burns, Paul Buckingham, Andreas Nickel, Daniel Macias-Castillo and others suggest that one should be able to

1. Remove the assumptions about real and complex embeddings and
2. Obtain a non-trivial annihilator from any single character and its Galois conjugates.
Invariants of $K$

- $U^S_K = "S\text{-units of } K" = \text{Units of } \mathcal{O}^S_K$
- $Y^S_K = \text{Free abelian group on the primes of } K \text{ above those in } S, \text{ a } \mathbb{Z}[G]\text{-module.}$
- $X^S_K = \text{Submodule of elements whose coefficients add up to 0.}$
- $R^S_K = "S\text{-regulator of } K" = \text{volume of a fundamental domain for } \mathbb{R} \otimes X^S_K \text{ modulo the image of } U^S_K \text{ under the logarithmic embedding.}$
- $h^S_K = "S\text{-class number of } K" = \text{order of } \text{Cl}^S_K.$
Let $\psi_0$ denote the trivial character of $G$. The lead term of the Taylor series for $L_S(s, \psi_0)$ at the origin is

$$L^*_S(\psi_0) = -\frac{h^S_F R^S_F}{w_F}.$$ 

Note that the appearance of $h^S_F$ in this formula suggests that an annihilator arises here regardless of assumptions on embeddings or the order of vanishing.
Choosing an Integral Homomorphism

For any $\mathbb{Z}$-module $A$, denote $\mathbb{R} \otimes \mathbb{Z} A$ by $\mathbb{R}A$.

- $\lambda : U_K^S \to \mathbb{R}X_K^S$, Dirichlet’s logarithmic map.
- $\lambda_{\mathbb{R}} : \mathbb{R}U_K^S \to \mathbb{R}X_K^S$, the induced isomorphism.
- $f : U_K^S \to X_K^S$, any chosen $\mathbb{Z}[G]$-module homomorphism with finite cokernel, which exists because the representations becoming isomorphic over $\mathbb{R}$ must be isomorphic over $\mathbb{Q}$.
- $f_{\mathbb{R}} \circ \lambda_{\mathbb{R}}^{-1}$, using the $\mathbb{R}$-linear extension of $f$, is an automorphism of the (necessarily projective) $\mathbb{R}[G]$-module $\mathbb{R}X_K^S$.
- $R(f) = \det_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda_{\mathbb{R}}^{-1})$, defined by extending by the identity on a complementary module $Z$ such that $(\mathbb{R}X_K^S) \oplus Z$ is free over $\mathbb{R}[G]$.
- $r(S, \psi) = \dim_{\mathbb{R}}(e_{\psi}\mathbb{R}X_K^S) = \text{ord}_{s=0}(L_S(s, \psi))$
Our Test Cases

Take

- $F$ a real quadratic field of discriminant up to 2000.
- $p$ a small prime congruent to 1 modulo 6 that splits into $p$ and $p'$ in $F$.
- $K$ an abelian, degree 6 extension of $F$ ramified only at $p$ and one infinite prime. (found using Stark’s classic rank-one conjecture)
- $\alpha$ an annihilator of $\mu_K = \{\pm 1\}$ in $\mathbb{Z}[G]$, e.g. 2 or $1 + \sigma$
- $\Phi_{K/F}(f) = R(f)\theta'_{K/F}(0)$ which is nonzero in $\mathbb{R}[G]$. ($f$-Stickelberger element)
- $\chi \in \hat{G}$ generator of order 6.
We verify the existence of a Stark unit generating each extension $K/F$ (125 examples, agreement to 40 digits).

$|G|\Phi_{K/F}(f) \in \mathbb{Z}[G]$ (39 examples with $h_K \neq 1$).

$|G|\Phi_{K/F}(f)(e_\chi + \bar{e}_\chi)$ annihilates $\text{Cl}_K$. (ditto)

$\alpha\Phi_{K/F}(f)(e_\chi + e_\chi^3 + \bar{e}_\chi)$ annihilates $\text{Cl}_K$. (ditto)
Further Details

- $\Phi_{K/F}(f)(e_\chi + e_{\bar{\chi}})$ has a denominator dividing 6, conjecturally. We found a denominator of 3 about 80% of the time, and otherwise 1.

- $\Phi_{K/F}(f)(e_\chi + e_{\chi^3} + e_{\bar{\chi}})$ has a denominator dividing 2, conjecturally. We found a denominator of 1 about 80% of the time, and otherwise 2.

- The annihilators we find often determine the action of $G$ on $\text{Cl}_K$ uniquely, although we do not generally find a full set of generators for the ideal of all annihilators.

- In most cases we consider, $\text{Cl}_K = \text{Cl}_K^S$. However, when they are not equal, we still find annihilators of $\text{Cl}_K$. 