L-functions at the origin and annihilation of S-class groups in multiquadratic extensions

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Brumer’s Conjecture: When $K/F$ is an abelian extension of number fields with Galois group $G$, Brumer’s conjecture states that the values at the origin of the Artin $L$-functions for $K/F$ can be used to provide annihilators in $\mathbb{Z}[G]$ of the ideal class group of $K$. 

Limitation: The annihilators obtained this way are 0 unless $F$ is totally real and $K$ is totally complex.

Key Question: Can one obtain non-trivial annihilators of class groups from the behavior of $L$-functions at the origin for all abelian extensions of number fields?
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S-Class Groups

- $F$ = algebraic number field (finite degree over the rationals).
- $K$ = finite abelian Galois extension of $F$
- $G$ = Galois group of $K$ over $F$
- $S$ = Finite set of primes of $F$ including all infinite primes and all primes ramifying in $K$
- $\mathcal{O}_S^K$ = “$S$-integers of $K$”, elements of $K$ whose valuation is non-negative at each prime not lying above a prime of $S$.
- $\text{Cl}_S^K$ = “$S$-class group of $K$”, nonzero fractional ideals of $\mathcal{O}_S^K$ modulo principal nonzero fractional ideals. It is a module over $\mathbb{Z}[G]$. 

Annihilating S-class groups
Artin L-Functions

- \( \psi \in \hat{G} = \text{group of characters of the abelian group } G = \text{Gal}(K/F). \)
- \( \sigma_p \in G \) is the Frobenius automorphism for a prime \( p \) of \( F \) that is not in \( S \) (therefore unramified in \( K \)).
- \( Np = \text{absolute norm of the ideal } p. \)
- \( L_S(s, \psi) = \prod_{\text{prime } p \notin S} \left(1 - \frac{1}{Np^s} \psi(\sigma_p)\right)^{-1} \) for \( \text{Re}(s) > 1 \), extends meromorphically to \( \mathbb{C} \).
The Stickelberger Element

- $e_\psi = \frac{1}{|G|} \sum_{\sigma \in G} \psi(\sigma)\sigma^{-1}$, the $\psi$-idempotent in $\mathbb{C}[G]$

- $\theta^S_{K/F}(0) = \sum_{\psi \in \hat{G}} L_S(0, \psi^{-1})e_\psi$, the $S$-Stickelberger element for $K/F$.

- $\theta^S_{K/F}(0) \in \mathbb{Q}[G]$, by Klingen-Siegel
Brumer’s Conjecture

- $\mu_K = \text{group of roots of unity of } K$, a module over $\mathbb{Z}[G]$.
- $w_K = |\mu_K|$, the order of this group.
- $\alpha = \text{any annihilator of } \mu_K \text{ in } \mathbb{Z}[G]$, e.g. $w_K$.
- $\alpha \theta^S_{K/F}(0) \in \mathbb{Z}[G]$ by Deligne-Ribet, Cassou-Nogues, Barsky.

Conjecture (Consequence of Brumer’s Conjecture)
$\alpha \theta^S_{K/F}(0)$ annihilates $\text{Cl}^S_K$.
Known to be true for $F = \mathbb{Q}$ by Stickelberger’s Theorem.
Observations

- Unless $F$ is totally real and $K$ is totally complex or $|S| = 1$, the previous conjecture is vacuous because $\theta_{K/F}^S(0) = 0$.
- Producing an annihilator involves using all $\chi$ for which $L_S(0, \chi) \neq 0$.
- David Burns, Paul Buckingham and others have suggested that one should be able to
  1. Remove the assumptions about real and complex embeddings and
  2. Obtain a non-trivial annihilator from any single character and its Galois conjugates.
Invariants of $K$

- $U^S_K = \text{"}S\text{-units of } K\text{"} = \text{Units of } \mathcal{O}_K^S$
- $Y^S_K = \text{Free abelian group on the primes of } K \text{ above those in } S, \text{ a } \mathbb{Z}[G]\text{-module.}$
- $X^S_K = \text{Submodule of elements whose coefficients add up to 0.}$
- $R^S_K = \text{"}S\text{-regulator of } K\text{"} = \text{volume of a fundamental domain for } \mathbb{R} \otimes X^S_K \text{ modulo the image of } U^S_K \text{ under the logarithmic embedding.}$
- $h^S_K = \text{"}S\text{-class number of } K\text{"} = \text{order of } \text{Cl}_K^S.$
Let $\psi_0$ denote the trivial character of $G$. The lead term of the Taylor series for $L_S(s, \psi_0)$ at the origin is

$$L_S^*(\psi_0) = -\frac{h_S^F R_F^S}{w_F}.$$ 

Note that the appearance of $h_S^F$ in this formula suggests that an annihilator arises here regardless of assumptions on embeddings.
Suppose that $\psi$ is a character of order 2 with fixed field $E$, so $[E : F] = 2$. Then the properties of $L$-functions and the analytic class number formula imply that the lead term of the Taylor series for $L_S(s, \psi)$ at the origin is

$$L_S^*(\psi) = \frac{h_E^S R_E^S w_F}{h_F^S R_F^S w_E}.$$
Choosing an Integral Homomorphism

For any $\mathbb{Z}$-module $A$, denote $\mathbb{R} \otimes_{\mathbb{Z}} A$ by $RA$.

- $\lambda : U^S_K \rightarrow \mathbb{R}X^S_K$, Dirichlet’s logarithmic map.
- $\lambda_R : \mathbb{R}U^S_K \rightarrow \mathbb{R}X^S_K$, the induced isomorphism.
- $f : U^S_K \rightarrow X^S_K$, any chosen $\mathbb{Z}[G]$-module homomorphism with finite cokernel, which exists because the representations becoming isomorphic over $\mathbb{R}$ must be isomorphic over $\mathbb{Q}$.
- $f_R \circ \lambda^{-1}_R$ is an automorphism of the (necessarily projective) $\mathbb{R}[G]$-module $\mathbb{R}X^S_K$.
- $R(f) = \det_{\mathbb{R}[G]}(f_R \circ \lambda^{-1}_R)$, defined by using the identity on $Z$ such that $(\mathbb{R}X^S_K) \oplus Z$ is free over $\mathbb{R}[G]$.
- $r(S, \psi) = \dim_{\mathbb{R}}(e_\psi \mathbb{R}X^S_K) = \text{ord}_{s=0}(L_S(s, \psi))$
Our Main Theorem

Theorem
Let $K$ be a composite of a finite number of quadratic extensions of a number field $F$. Let $S$, $f$, $\alpha$ and $\psi$ be as above. Then $|G|^{r(S,\psi)+1} \alpha R(f)L^*_S(\psi)e_\psi$ lies in $\mathbb{Z}[G]$ and annihilates $\text{Cl}_K^S$. 
Remark

When $T$ is a set of primes disjoint from $S$, David Burns works instead with the group of $S$-units which are congruent to 1 modulo primes above those in $T$, and with the correspondingly modified $L$-series. A theorem for multiquadratic extensions analogous to the one above follows from his results, but it introduces an extra factor of $|G|^2$. Burns’ student Daniel Macias Castillo strengthens this result, under the assumption that the character is one for which the order of vanishing of the corresponding $L$-function at the origin is minimal.