Before we do physics, we must learn some mathematics. The first area of mathematics which we will look at (and never really stop looking at) is called analytical geometry.

In analytical geometry, statements about lines and curves are expressed as statements about algebraic equations, and vice versa. We do this in the plane by first laying down two straight lines, which are perpendicular to each other, called the x-axis and the y-axis. With these axes we can assign a pair of numbers to every point in the plane. We simply treat the axes as two number lines, each with the same scale and set so that their two zeroes coincide. See the diagram to the right. The point that is a unit to the right of the y-axis and b units above the x-axis has assigned to it the pair (a, b). This is called an ordered pair; because if a ≠ b, then (a, b) ≠ (b, a); the order is important.

If a point's pair is (a, b), then a is called its x-coordinate or abscissa, and b is its y-coordinate or ordinate. Points to the left of the y-axis have negative abscissas; points below the x-axis have negative ordinates.

We can define a set of points in the plane by saying that a point belongs in this set (call it $S$) if and only if its x- and y-coordinates satisfy some relation. We could let $S$ be the set of all points (a, b) such that $a^2 + b^2 = 2$.

We can abbreviate that last statement as:

$$S = \{ (a, b) : a^2 + b^2 = 2 \}$$

That is how we will abbreviate all such definitions henceforward.

Our set $S$ is a circle of radius $\sqrt{2}$, centered around the point (0, 0) (called the origin). We can see why this is so, because of the Pythagorean theorem. Pictures (or graphs) of other point-sets, with their definitions, are given below.

We are most interested in those point-sets in which for each real number $a$, there is at most one point in the set with abscissa equal to $a$. In other words, any line I draw in the plane, which is perpendicular to the x-axis, will intersect such a set in at most one point. For a point in this set, we say that its y-coordinate is a function of its x-coordinate, or "$y$ is a function of $x" or (more abbreviated still): $y = f(x)$. The set, or its graph, represents the function—which is the abstract relation between the ordinates and abscissas.
The statement "$y = f(x)$" means: given a number $x$, I can find a unique $y$, or no $y$ which corresponds to $x$. One way to get one number from another is to use the algebraic operations: addition, multiplication, etc. Some things which $f(x)$ might be are:

$$f(x) = x^n + 1$$

$$f(x) = (2x^2 - 3x^2)/(x^3 + 3)$$

$$f(x) = 2x - 3$$

A set representing the first function would be one in which a point's ordinate was equal to its abscissa squared, plus one.

A particularly simple-minded type of function is one whose point-set is a straight line. Such a thing is called an affine function. All affine functions can be written in the form:

$$f(x) = mx + b$$

where $m$ and $b$ are fixed real numbers. The graphs of three affine functions are given below:

$$y = x$$

$$y = 2x$$

$$y = \frac{x}{2} + 2$$

The $m$ in (*) is called the slope of the line $y = mx + b$. Given any two points on that line—say $(a, b)$ and $(c, d)$—the quantity:

$$m = \frac{b - d}{a - c}$$

always equals $m$. Work it out, if you don't believe it.

**Homework:** (Do all homework on another sheet of paper.)

1. What is the affine function whose line goes through the points:
   a. (1, 2) and (3, 7)?
   b. (0, 0) and (4, 25)?
   c. (2, 1) and (16, 0)?

2. Which of the following relations between $x$ and $y$ are functions? ($y = f(x)$)
   a. $x + y = 1$.
   b. $2x + 3y = 7$.
   c. $x^2 - y = 0$.
   d. $x^2 - y^2 = 0$.
   e. $x - y^2 = 0$.

3. Does a straight line go through the points (0, 0), (3, 4), and (5, 7)?

A very important set in the plane (which does not represent a function) is the set:

$$\{(x, y) : x^2 + y^2 = 1\}$$

It is the circle of radius one, centered at the origin, and it is called the unit circle. If we draw a ray out from the origin, it will make some angle (measured counterclockwise) with the positive half of the $x$-axis, as in the diagram below:
The ray will intersect the unit circle in one point, say \((a, b)\). The number \(a\) is called the cosine of the angle \(\theta\) (theta; pronounced "thayta"); \(b\) is the sine of \(\theta\). We abbreviate these relations as:

\[
\begin{align*}
\cos \theta &= a \\
\sin \theta &= b
\end{align*}
\]

"Notice that, from the definition, \((\cos \theta)^2 + (\sin \theta)^2 = 1\), for any angle \(\theta\). From now on, instead of \((\cos \theta)^n\) and \((\sin \theta)^n\), we will write—as per tradition—\(\cos^n \theta\) and \(\sin^n \theta\).

Although the unit circle does not represent a function, sine and cosine are functions. From them, we can get more functions:

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta}
\]

\[
\cot \theta = \frac{1}{\tan \theta} \quad \sec \theta \tan \theta = \csc^2 \theta
\]

We can derive the following equalities—I will prove some in class, and leave the rest for homework:

\[
\begin{align*}
\tan^2 \theta + 1 &= \sec^2 \theta \\
\cot^2 \theta + 1 &= \csc^2 \theta
\end{align*}
\]

\[
\cos 45^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}} \quad \tan 90^\circ = \infty = \text{infinity}
\]

\[
\cos(90^\circ - \theta) = \sin \theta \quad \sin(90^\circ - \theta) = \cos \theta
\]

\[
\begin{align*}
\sin(\theta_1 + \theta_2) &= \sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1 \\
\cos(\theta_1 + \theta_2) &= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2
\end{align*}
\]

\[
\begin{align*}
\tan \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{2}} \\
\sin \frac{\theta}{2} &= \sqrt{\frac{1 - \cos \theta}{2}}
\end{align*}
\]

\[
\begin{align*}
\sin \theta_1 - \sin \theta_2 &= 2 \cos \left(\frac{\theta_1 + \theta_2}{2}\right) \sin \left(\frac{\theta_1 - \theta_2}{2}\right) \\
\cos \theta_1 - \cos \theta_2 &= -2 \sin \left(\frac{\theta_1 + \theta_2}{2}\right) \sin \left(\frac{\theta_1 - \theta_2}{2}\right)
\end{align*}
\]

\[
\begin{align*}
\sin(-\theta) &= -\sin \theta \\
\cos(-\theta) &= \cos \theta
\end{align*}
\]
1. Consider the circle \( C = \{(x,y) : (x-a)^2 + (y-b)^2 = r^2 \} \). Suppose that the point \((c,d)\) is on the circle, with \(d > b\). What are the coordinates of the point on the circle that is directly below \((c,d)\)?

2. Let \( x \) be an angle between 0 and \( \pi/4 \). Suppose that \( \sin x = a \) and \( \cos x = b \).

In the diagram below, if \( \theta = 3x/2 \), and the triangle's base length is \( L \), what is its height \( H \) in terms of \( a \), \( b \), and \( L \)?

\[ \text{Height} = \frac{a \cdot b}{L} \]

3. Use the addition formulas for the sine and cosine to prove that

\[ \tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \cdot \tan \theta_2} \]

4. Consider the line represented by \( y = mx + b \), where \( m \neq 0 \). Find the affine function for the line which is perpendicular to this one, and which goes through the point \((r, mr + b)\).

5. What are the coordinates of the point where the curves of:

\[
\begin{align*}
y &= x^2 + 3 \\
y &= 2x + 2
\end{align*}
\]

intersect?

6. In the diagram to the right, \( \theta \) is between 0 and \( \pi \).

What is the area of the shaded region, if the circle's radius is one, and \( \theta \) is given in radians?
**Limits**

Let \( f(x) \) be a function of \( x \). If, as we take values of \( x \) closer and closer to some number (which may be infinite), \( f(x) \) gets closer and closer to a particular number, \( f(x) \) is said to approach a limit. With more rigor, we have the following two definitions of "limit":

I. Let \( A \) be a fixed real number. If for every small number \( \varepsilon > 0 \), there exists a large number \( N \) such that \( x > N \) implies:

\[
|f(x) - A| < \varepsilon
\]

then we say that \( A \) equals the limit of \( f(x) \) as \( x \) goes to infinity.

We abbreviate this as:

\[
\lim_{x \to \infty} f(x) = A
\]

II. Let \( A \) and \( a \) be fixed real numbers. If for every small number \( \varepsilon > 0 \), there exists another small number \( \delta > 0 \), such that:

\[
0 < |x - a| < \delta
\]

implies:

\[
|f(x) - A| < \varepsilon
\]

then we say that \( A \) equals the limit of \( f(x) \) as \( x \) goes to \( a \), and we abbreviate this as:

\[
\lim_{x \to a} f(x) = A
\]

Definition I says that as \( x \) gets larger and larger ("closer to infinity"), \( f(x) \) gets close to \( A \). The function gets as close to \( A \) as we like, as \( x \) becomes large—which is why we can say that \( f(x) \) "goes to" \( A \). The distance between the value of \( f(x) \) and \( A \) eventually—as \( x \) gets big—becomes smaller than any positive number we can pick. The diagram below shows this.

---

The function is \( 1/x \). If \( x \) is greater than 15, then \( 1/x = |1/x - 0| < 1/15 \). No matter how small \( \varepsilon \) is, we can always find a number so large that if \( x \) is bigger than it, then \( 1/x \) is smaller than \( \varepsilon \). Therefore, we say that \( 1/x \) goes to 0 as \( x \) goes to infinity. (What is the limit of \( f(x) = 1/x + 3 \) as \( x \) goes to infinity?)

Definition II concerns limits in which \( x \) approaches a finite number. It says, roughly, that if \( x \) isn't very far from \( a \) (\( |x - a| < \delta \)), then \( f(x) \) isn't very far from \( A \) (\( |f(x) - A| < \varepsilon \)). This is like Definition I, which says that if \( x \) is very large—greater than \( N \)—then \( f(x) \) is close to \( A \) (\( |f(x) - A| < \varepsilon \)).
In the diagram above, \( f(x) \) is \( 1/x \). If \( x \) is within \( 1/10 \) of 2, then \( f(x) \) is within \( 1/50 \) of \( 1/2 \). Here our \( \delta \) is \( 1/50 \) and our \( \epsilon \) is \( 1/10 \).

If \( \epsilon = 1/1000000 \), then a \( \delta \) that will work is \( 2/1000000 \); if the distance from \( x \) to 2 is less than \( 2/1000000 \), then the distance from \( 1/x \) to \( 1/2 \) is less than \( 1/1000000 \). Since we can do this indefinitely, making \( \epsilon \) as small as we like, we say that \( 1/2 \) is the limit of \( 1/x \) as \( x \) goes to 2.

The following facts about limits are useful. I may prove a couple of them. They should seem obvious.

Let \( a \) be a real number (possibly infinite). Let \( f(x) \) and \( g(x) \) be two functions, with \( A \) and \( B \) two real numbers such that:

\[
\lim_{x \to a} f(x) = A \quad \lim_{x \to a} g(x) = B
\]

(Neither \( A \) nor \( B \) can be infinite. Why?)

Then:

\[
\lim_{x \to a} (f(x) \pm g(x)) = A \pm B
\]

\[
\lim_{x \to a} (f(x) \cdot g(x)) = A \cdot B
\]

\[
\lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = \frac{A}{B} \quad \text{if} \quad B \neq 0.
\]

One usually does limits by means of a trick; for example, by multiplying the top and bottom of an expression by something which makes the limit obvious. To find the limit of:

\[
\frac{\sqrt{x} - 1}{x - 1}
\]

as \( x \) goes to 1, we note that for \( x \neq 1 \), this expression is equal to:

\[
\frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{1}{\sqrt{x} + 1} = \frac{1}{\sqrt{x} + 1}
\]

As \( x \) approaches 1, this approaches 1/2. Therefore, the limit is 1/2.

We do a similar thing to find the limit of:

\[
\frac{1 - \sin x}{\cos x}
\]

as \( x \) approaches \( \pi/2 \). For \( x \neq \pi/2 \), the expression is 0/0—nonsense! But, remembering that \( 1 - \sin^2 x = \cos^2 x \), we see that for \( x \) not equal to \( \pi/2 \), our function equals:

\[
\frac{1 - \sin x}{\cos x} \cdot \frac{1 + \sin x}{1 + \sin x} = \frac{1 - \sin^2 x}{\cos x} \cdot \frac{1}{1 + \sin x} = \frac{\cos x}{1 + \sin x}
\]

Therefore, the limit is 0.

The limit, as \( x \) goes to infinity, of \( (x^3 + 3x)/(9x^3 - 7) \) is found by multiplying the top and bottom of the expression by \( 1/(x^3) \). It becomes:

\[
\left( 1 + 3x^2 \right) \left( \frac{9}{x^3} \right)
\]
which obviously goes to 1/9.

**Homework**: Find these limits.

1. \( \lim_{x \to 1} \frac{x^n - 1}{x - 1} \) (n a positive integer)

2. \( \lim_{x \to \infty} \sqrt[3]{\frac{1}{a} - x} \) (a a fixed real number)

3. \( \lim_{x \to \pi/4} \frac{\cos x - \sin x}{\cos 2x} \)

4. \( \lim_{x \to 0} \frac{\sin x}{\tan x} \)

5. \( \lim_{x \to 0} \frac{\sin 2x}{\sin x} \)

6. \( \lim_{x \to \pi/4} \frac{\cos x - \sin x}{1 - \tan x} \)

7. Is there a number \( A \) such that: 1) \( \lim_{x \to 0} \frac{\sin (1/x)}{x} = A; \) 2) \( \lim_{x \to 0} \frac{x \sin (1/x)}{x} = A. \) If so, find the number(s) it.
1. Find these limits.
   a. \( \lim_{x \to 0} \frac{5x}{x} \)
   b. \( \lim_{x \to a} \frac{(\sin x - \sin a)}{\cos x - \cos a} \); where \( a \) is an angle between 0 and \( \frac{\pi}{2} \) radians.
   c. \( \lim_{x \to 0} (x + 3) \)
   d. \( \lim_{x \to 2} \frac{(x^2 - 3x + 2)}{(x^2 - 6x + 8)} \)

2. Do these limits exist? Just write yes or no.
   a. \( \lim_{x \to \infty} \sin x \)
   b. \( \lim_{x \to \infty} \frac{(\sin x)}{x} \)
   c. \( \lim_{x \to \frac{\pi}{2}} \sec x - \tan x \)
   d. \( \lim_{x \to 0} \frac{(1 - \cos x)}{(\sin^3 x)} \)
Derivatives

The first part of calculus—called differential calculus—is concerned with how one finds the slope of a curve at a point. If the curve is a straight line, then this is easy: the slope at any point is just the slope of the line. But for functions which are not affine, the problem is trickier. The diagram to the right is a drawing of the graph of the function \( y = x^2 \). What is the slope of the curve when \( x = 2 \)?

To answer that, we must first know what "slope of the curve" means. Since a parabola has no uncurved segments like a line, it isn't obvious what we should call the steepness of the curve at a point.

The meaning which mathematicians have given to "slope of the curve" is: the slope of the line that is tangent to the curve at that point. In our diagram, the line tangent at \( (2, 4) \) is \( y = 4x - 4 \).

Why is this line called the tangent line? A line is said to be tangent to a circle if it intersects it in only one point. For any point on a circle, there is only one line which is tangent to the circle at that point: just one line which intersects at only that point. Now, this isn't true of the line in our diagram. A vertical line passing through \( (2, 4) \) also intersects the curve in only one point.

The line \( y = 4x - 4 \) is called a tangent because, if we change its slope a little (while making it still pass through \( (2, 4) \)), it will intersect the curve in more than one point. (Just as a tangent to a circle, if turned a little about its tangent point, will intersect the circle in another point.) And not just "one point," but a point that is very close to \( (2, 4) \). If our curve had been like the one on the left, which is like \( y = x^2 \) when \( x \) is less than \( 2 \), and then curves down, then the line \( y = 4x - 4 \) would still be a tangent at \( (2, 4) \), although it intersects the curve somewhere else. Because, if we look at a part of the line which is close enough to \( (2, 4) \)—say, when \( x \) is between 1.9 and 2.1—then that part doesn't intersect the curve.

We find this tangent line by looking at lines which are "almost" tangent. In the next diagram we see several such lines. The line \( y = 5x - 6 \) intersects the parabola in two points: \( (2, 4) \) and \( (3, 9) \). So does the line \( y = (9/2)x - 5 \): at \( (2, 4) \) and \( (5/2, 6_2) \). Notice that \( (5/2, 6_2) \) is closer to \( (2, 4) \) than \( (3, 9) \) is. A line with a slope of \( 4_2 \), which passed through \( (2, 4) \), would intersect the curve at a point even closer to \( (2, 4) \). And so on. As the slopes of the lines—which pass through \( (2, 4) \)—approached \( 4_2 \), the two points at which they intersect the curve would get closer together. As a limit, they would become one point: \( (2, 4) \).

The same thing would happen if we looked at lines with slopes less than \( 4_2 \). As their slopes increased to \( 4_2 \), their intersection points would approach each other, to meet at \( (2, 4) \).
The slope of the tangent line is equal to the limit of the slopes of the "almost" tangent lines, taken as the lines' intersection-points approach each other. We can see that it is reasonable for these slopes to approach \( h \), by looking at the following table:

<table>
<thead>
<tr>
<th>POINT ON THE PARABOLA</th>
<th>SLOPE OF THE LINE THAT GOES THROUGH THAT POINT AND THE POINT ((2, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 0))</td>
<td>( \frac{2}{3} )</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>( 3.5 )</td>
</tr>
<tr>
<td>((1.5, 2.25))</td>
<td>( 3.75 )</td>
</tr>
<tr>
<td>((1.75, 3.0625))</td>
<td>( 4.25 )</td>
</tr>
<tr>
<td>((2.25, 5.0625))</td>
<td>( \frac{4}{1} )</td>
</tr>
<tr>
<td>((2.41, 4.41))</td>
<td></td>
</tr>
</tbody>
</table>

We find the slope of the tangent line by taking the limit:

\[
\lim_{x \to 2} \left( \frac{x^2 - 2^2}{x - 2} \right)
\]

which equals \( h \).

If our function had been some other \( f(x) \), we could have found its slope (or derivative) at a point \((a, f(a))\) by taking the limit:

\[
\lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \right)
\]

This number, when it exists, is called the derivative of \( f(x) \) at \( a \), and we denote it by: \( f'(a) \). That is what we found in (*) above. There \( f(x) = x^2 \), and \( a = 2 \).

**HOMEWORK:** Find \( f'(a) \) (\( a \) is a fixed real number) when \( f(x) \) equals:

1. \( x^3 \)
2. \( x^6 \)
3. \( mx + b; \) \( m \) and \( b \) are fixed real numbers
4. \( \frac{1}{(x^2)} \)
5. \( \frac{1}{(x^2 + 1)} \)
6. \( x^3 + 2x^2 - 12x + 5 \)
7. \( x/(x + 1) \)
8. \( 3 \)
9. \( \frac{c}{(c \text{ is a real number})} \)

(Note: Even if you know calculus, work these by finding the limit (**) for each of them. Show your work, or no credit. And be neat!)
More Derivatives

It is inconvenient to always have to work out the limit:

$$\lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \right)$$

whenever we want to find the derivative of a function. Therefore, we will learn three handy rules: the product rule, the quotient rule, and the chain rule. They will make finding derivatives as easy as falling off a logarithm.

I. Product Rule. Let $f(x)$ and $g(x)$ be functions that have derivatives at $a$. Define a function $h(x)$ by: $h(x) = f(x)g(x)$. Then $h(x)$ also has a derivative at $a$, and:

$$h'(a) = f'(a)g(a) + g'(a)f(a)$$

Proof: By definition, $h'(a)$ equals:

$$\lim_{x \to a} \left( \frac{h(x) - h(a)}{x - a} \right)$$

which is the limit, as $x$ goes to $a$, of:

$$\frac{f(x)g(x) - f(a)g(a)}{x - a}$$

which is the same (when $x \neq a$) as:

$$\frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a}$$

Separate this into two parts.

$$\frac{f(x)g(x) - f(a)g(x)}{x - a} + \frac{f(a)g(x) - f(a)g(a)}{x - a}$$

$$= g(x) \left( \frac{f(x) - f(a)}{x - a} \right) + f(a) \left( \frac{g(x) - g(a)}{x - a} \right)$$

As $x$ approaches $a$, the quantity on the left approaches $g(a)f'(a)$. The one on the right goes to $f(a)g'(a)$. Therefore:

$$h'(a) = f'(a)g(a) + g'(a)f(a)$$

Q.E.D.

II. Quotient Rule. Let $f(x)$ and $g(x)$ be as in I., but also stipulate that $g(a) \neq 0$. Let $j(x) = f(x)/g(x)$. The function $j(x)$ has a derivative at $a$, and:

$$j'(a) = \left( \frac{f'(a)g(a) - g'(a)f(a)}{(g(a))^2} \right)$$
Proof: The definition of the derivative says that:

\[ f'(a) = \lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \right) \]

The expression inside the parentheses equals (when \( x \neq a \)):

\[ \frac{f(x)g(a) - f(a)g(x)}{g(a)g(x)} = \]

A trick similar to what we did in I. turns this into:

\[ = \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{g(a)g(x)} \]

We break this up into two parts, and bring \( g(a)g(x) \) outside to get:

\[ = \frac{1}{g(a)g(x)} \left( f(x)g(a) - f(a)g(a) \right) + \frac{f(a)g(a) - f(a)g(x)}{x - a} \]

\[ = \frac{1}{g(a)g(x)} \left( g(a) \left( f(x) - f(a) \right) \right) + \frac{f(a) \left( g(x) - g(a) \right)}{x - a} \]

As \( x \) approaches \( a \), the quantity marked A goes to \( 1/(g(a))^2 \). The quantity B approaches \( g(a)g'(a) \); C approaches \( f(a)g'(a) \). Therefore, the whole thing goes to:

\[ \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2} \]

And this is what we were trying to prove.

III. Chain Rule. Let \( g(x) \) have a derivative at \( x = a \), and let \( f(x) \) be a function that has a derivative at \( x = g(a) \); i.e., \( f(x) \) is such that:

\[ f'(g(a)) \]

exists. Let \( k(x) = f(g(x)) \). Then \( k'(a) \) exists, and it equals:

\[ g'(a)f'(g(a)) \]

Proof: From the definition, \( k'(a) \) equals:

\[ \lim_{x \to a} \left( \frac{f(g(x)) - f(g(a))}{x - a} \right) \]

When \( x \neq a \), the expression in the parentheses can be written as:

\[ \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \neq \frac{g(x) - g(a)}{x - a} \]
Now, look at the expression on the left. As we make \( x \) approach \( a \),
g(\( x \)) must approach \( g(a) \); and it must approach \( g(a) \) smoothly, since
the function \( g(x) \) is continuous near \( a \)(otherwise, it wouldn't have
a derivative there). Because of this, we can treat \( g(x) \) as if it
were just a variable. As \( x \) goes to \( a \), the quotient on the left
approaches:

\[
\lim_{x \to a} \left( \frac{f(x) - f(g(a))}{x - g(a)} \right)
\]

This limit equals \( f'(g(a)) \).

As \( x \) goes to \( a \), the expression becomes \( g'(a) \). Taking their product, we get:

\[
k'(a) = g'(a) \times f'(g(a))
\]

Q.E.D.

Example: Let \( h(x) = x^2 \sin x \). What is \( h'(x) \)? Use the product rule, letting
\( f(x) = x^2 \), \( g(x) = \sin x \). Then \( h'(x) \) equals:

\[
2x \sin x + x^2 \cos x
\]

Example: Let \( r(x) = \tan x \). What is \( r'(x) \)? Use the quotient rule. Since
\( r(x) = \sin x / \cos x \), let \( f(x) = \sin x \) and \( g(x) = \cos x \). The rule gives:

\[
\frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}
\]

\[
= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}
\]

\[
= \frac{1}{\cos^2 x} = \sec^2 x
\]

Homework: Find the derivatives of these functions.

1. \( \tan^2 x \)
2. \( \tan x \)
3. \( \tan^k x \) (\( k \) is a positive integer)
4. \( x/(x - \sin^2 x) \)
5. \( \sin(x^2) \)
6. \( \sin(\cos x) \)
7. \( (x + \cos x)^3 \)
8. \( 1/(x^2 + 1) \)
9. \( \tan(1/(x^2 + 1)) \)
10. \( 1/(\tan x + \sin x + \cos x) \)
1. Let \( f(x) = |x| \). Prove that \( f'(0) \) does not exist (prove it, don't just draw a picture).

2. Find the derivatives of these functions.
   
   a. \( \frac{1}{(\sec^2 x - 1)} \)
   
   b. \( \frac{\tan 3x}{\tan hx} \)
   
   c. \( (x^2 + 1)^n \) \((n \text{ is a positive integer})\)
   
   d. \( \frac{x^2 - 6x + 8}{x^2 - 5x + 6} \)
   
   e. \( \cot x \)
   
   f. \( \frac{1}{x - x^2} \)
   
   g. \( \cos^2 x + \sin^2 hx \)
   
   h. \( \frac{x^2 + 1}{x^2 - 1} \)

3. Suppose that \( f(x) \) and \( g(x) \) are two functions, and that \( f(x) = (x-a)g(x) \), where \( a \) is a real number. Use the product rule to find \( f'(a) \).
Integration

The second part of calculus is called integral calculus. In differential calculus, we are concerned with finding the slope of a curve at a point. In integral calculus, we are looking for the area which is under a curve. To integrate a function means, roughly, to find the area under that function's curve. The integral of a function is that area. But what precisely do we call the "area?" We usually define the integral in terms of the endpoints of some interval on the x-axis. In the diagram to the left, the interval is $[a, b]$. The area of the shaded region is called the integral of $f(x)$ from $a$ to $b$. We denote this number with the symbol:

$$\int_{a}^{b} f(x) \, dx$$

When $f(x)$ goes below the x-axis, it contributes negative area to the integral. In our diagram, the cross-hatched region has negative area.

If the number $(#)$ exists, then $f(x)$ is said to be integrable on $[a, b]$. If we make $a$ and $b$ be finite numbers, then most functions you can think of will be integrable on $[a, b]$. This isn't true if $a$ or $b$ becomes infinite. The integral:

$$\int_{0}^{\infty} \sin x \, dx$$

(which is the area under the curve $\sin x$ for all $x$ greater than 0) doesn't exist. Can you see why? This integral:

$$\int_{1}^{\infty} \frac{1}{x} \, dx$$

(the area under $y = 1/x$ for all $x$ greater than 1) is infinite. We'll see why later.

These facts about integrals are useful:

1. If $f(x)$ and $g(x)$ are integrable on $[a, b]$, then:

   $$\int_{a}^{b} [f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx$$

2. If $f(x)$ is integrable on $[a, b]$, and $c$ is any constant, then:

   $$\int_{a}^{b} c \cdot f(x) \, dx = c \int_{a}^{b} f(x) \, dx$$

3. If $a < b < c$, and $f(x)$ is integrable on $[a, b]$ and $[b, c]$, then $f(x)$ is also integrable on $[a, c]$, and:

   $$\int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx = \int_{a}^{c} f(x) \, dx$$

The meaning of $(3)$ is shown in the picture below.
1. The integral: \( \int_a^q f(x) \, dx \) equals zero, for any real number \( a \).

I will go over these facts in class.

In (*), the function is integrated from \( a \) to \( b \), where \( a \leq b \). Is it possible to integrate backwards, from \( b \) to \( a \)? Yes. And the number we get--the integral of \( f(x) \) from \( b \) to \( a \)--denoted: \( \int_b^a f(x) \, dx \) is the negative of (*). We can see this by assuming the following generalization of fact (3.):

3' If \( a, b, c \) are any numbers, and \( f(x) \) is integrable on the appropriate intervals, then:

\[
\int_a^c f(x) \, dx + \int_b^c f(x) \, dx = \int_a^b f(x) \, dx
\]

Now, let \( a, b, \) and \( c \) be our three numbers. The above assumption gives:

\[
\int_a^c f(x) \, dx + \int_b^c f(x) \, dx = \int_a^b f(x) \, dx
\]

But, by fact (1'), we know that: \( \int_a^b f(x) \, dx = 0 \). Therefore,

\[
\int_a^c f(x) \, dx = -\int_b^c f(x) \, dx
\]

which is what we claimed.

Some Simple Integrals.

I. Let \( f(x) = c \), a constant. What is \( \int_b^a f(x) \, dx \)? If \( a \leq b \), then the integral is the area of rectangle ABCD, which is \((b-a) \times c\).

II. Let \( f(x) = mx \), \( m \) a constant. The integral of \( f(x) \) from \( a \) to \( b \) (again, assuming that \( a \leq b \)) is the area of the trapezoid PQRS. This equals \( \frac{1}{2} (ma + mb) \times (b-a) = \frac{1}{2} m(a+b) \times (b-a) = \frac{1}{2} m(b^2 - a^2) \).

III. Let \( f(x) = x^2 \). What is \( \int f(x) \, dx \)? Here we must be tricky—as Archimedes was. Below are two sketches of the graph of \( f(x) \). We have covered each drawing with a lot of rectangles. In the drawing on the left, the rectangles are slightly higher than the curve. The sum of the areas of these rectangles is a little larger than the area under the curve. The rectangles on the right are lower than...
the curve; the sum of their areas is less than area under the curve. We have this inequality:

\[
\text{area of lower rectangles} \leq \int_0^1 x^2 \, dx \leq \text{area of higher rectangles}
\]

As we make the rectangles thinner and thinner and let their number increase without limit, the area of the "lower" rectangles will approach the area of the "higher" rectangles, as a limit. The value of the integral will be squeezed between them. So to know the integral, we have to find the limit of the rectangles' areas. Since the limits of the two sets of rectangles are equal, we'll find it for one set—say, the higher ones.

We do this by dividing the interval \([0,1]\) into \(n\) (where \(n\) is some large natural number) segments of equal length. These segments will be the bases of our rectangles. Next we draw our rectangles—starting from these bases—just tall enough so that their right corners touch the curve \(y = x^2\). See the diagram to the left.

A typical rectangle will have its base between the points \((k-1)/n\) and \(k/n\), where \(k\) is a natural number between 1 and \(n\) (inclusive). So the length of any rectangle’s base will be equal to: \(\frac{k}{n} - \frac{k-1}{n} = \frac{1}{n}\).

Its height will be the height of the point at which it touches the curve. From the diagram, we see that this is where \(x = k/n\), or \(y = k^2/n^2\). Therefore, the rectangle’s height is \(k^2/n^2\), and its area—base times height—is \(k^2/n^3\).

The sum of the areas of the rectangles equals:

\[
\frac{1^2}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \cdots + \frac{n^2}{n^3}
\]

\[
= \frac{1}{n^3} \left( 1^2 + 2^2 + 3^2 + \cdots + n^2 \right)
\]

\((***)\)

A handy formula (which I will prove and discuss in class, for those who haven't seen it) says that:

\[
\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}
\]

is equal to: \(n(n+1)(2n+1)/6\), for any \(n\). Substituting this into equation (***) yields:

\[
\frac{1}{n^3} \left[ \frac{n(n+1)(2n+1)}{6} \right]
\]

\[
= \frac{1}{n^3} \left[ \frac{2n+1}{6} \right] \times \frac{n(n+1)}{1+1/n}
\]

The integral is the limit of this expression as \(n\) goes to infinity—as the rectangles become thinner and more numerous, in other words. This limit is \(1/3\). Therefore:

\[
\int_0^1 x^2 \, dx = \frac{1}{3}
\]
Using this method, we can (theoretically) find the integral of almost any function we will work with, over any finite interval. But we won't. Integrating with rectangles all the time is like always differentiating by taking the limit of \((f(x) - f(a))/(x-a)\). Instead, we will use a powerful (and famous) theorem.

**The Fundamental Theorem of Calculus.**

The Fundamental Theorem says that integration is the opposite of differentiation, in the sense in which division is the opposite of multiplication. The derivative of a function's integral is equal to the function; and the integral of a function's derivative is "almost" equal to the function. More precisely, the two parts of the Fundamental Theorem are:

**Continuous**

I. Let \(f(x)\) be a function, and define a function \(G(x)\) by:

\[
G(x) = \int_a^x f(t) \, dt
\]

where \(a\) is a fixed real number. Then, wherever \(G'(x)\) exists:

\[
G'(x) = f(x)
\]

II. Let \(G(x)\) be a function such that \(G'(x) = f(x)\). Then, whenever the integral below exists, the equality:

\[
\int_a^b f(x) \, dx = G(b) - G(a)
\]

holds.

First we prove I. Then we use I. to prove II.

**Proof of I.** We want to show that, given an \(x_0\) for which \(G'(x_0)\) exists:

\[
(\#) \quad f(x_0) = \lim_{x \to x_0} \frac{G(x) - G(x_0)}{x - x_0}
\]

This limit can only exist if \(f(x)\) is continuous at \(x_0\). To see this, consider the diagram below. Here, \(f(x) = 0\) for \(x < x_0\), and equals 1 for \(x > x_0\). The graph of this function's \(G(x)\) is to the right. It has a corner at \(x_0\): no derivative! Therefore, we can assume that \(f(x)\) is continuous at \(x_0\); i.e., that \(\lim f(x) = f(x_0)\).

Saying that the limit in (\#) equals \(f(x_0)\) is the same as saying that, as \(x\) approaches \(x_0\), the distance between \(f(x_0)\) and the quotient on the right approaches zero. In other words, (\#) will be true if and only if:

\[
(\#\#) \quad 0 = \lim_{x \to x_0} \left| \frac{G(x) - G(x_0)}{x - x_0} - f(x_0) \right|
\]

We shall prove (\#\#). Now, \(G(x) - G(x_0) = \int_a^x f(t) \, dt - \int_a^{x_0} f(t) \, dt = \left[ \text{by what we've said earlier} \right] \int_a^{x_0} f(t) \, dt + \int_a^x f(t) \, dt = \int_a^{x_0} f(t) \, dt + \int_a^x f(t) \, dt = \int_a^x f(t) \, dt \). Therefore, we can rewrite (\#\#) as:

\[
(\#\#) \quad \lim_{x \to x_0} \left| \int_a^{x_0} f(t) \, dt + \int_a^x f(t) \, dt - f(x_0) \right|
\]
\[
\lim_{x \to x_0} \left| \frac{1}{x-x_0} \int_{x_0}^{x} f(t) \, dt - f(x_0) \right|
\]

Since \( f(x_0) \) is a constant, the integral: \( \int_{x_0}^{x} f(x_0) \, dt \) equals \( f(x_0)(x-x_0) \).

Then when \( x \neq x_0 \), we have that:

\[
f(x_0) = \frac{1}{x-x_0} \int_{x_0}^{x} f(t) \, dt
\]

We can substitute this for \( f(x_0) \) in our limit to get:

\[
\lim_{x \to x_0} \left| \frac{1}{x-x_0} \int_{x_0}^{x} f(t) \, dt - \int_{x_0}^{x} f(x_0) \, dt \right|
\]

\[
= \lim_{x \to x_0} \left| \frac{1}{x-x_0} \int_{x_0}^{x} [f(t)-f(x_0)] \, dt \right|
\]

A fact which I didn't put in this handout—and which I hope is obvious—is that if \( f(x) \leq g(x) \) for all \( x \) in \([a,b]\), then:

\[
\int_{x_0}^{x} f(x) \, dx \leq \int_{x_0}^{x} g(x) \, dx
\]

For any \( t \) at all, \( f(t)-f(x_0) \leq |f(t)-f(x_0)| \cdot \). Therefore, if we can show that our limit goes to zero, with \( \Delta \) replaced by the integral:

\[
\int_{x_0}^{x} |f(t)-f(x_0)| \, dt
\]

then we will be done. Now we use the fact that \( f(x) \) is continuous at \( x_0 \).

For any small number \( \varepsilon > 0 \), we can find a \( \delta > 0 \) so that \( 0 < |t-x_0| < \delta \) implies that \( |f(t)-f(x)| < \varepsilon \). If we can show that our limit is less than any such \( \varepsilon \)—no matter how small \( \varepsilon \) is—then that will mean that it must be zero.

So let's suppose we've been given an \( \varepsilon \). Choose the "appropriate" \( \delta \).

For any \( x \) such that \( 0 < |x-x_0| < \delta \), we have:

\[
\left| \int_{x_0}^{x} |f(t)-f(x_0)| \, dt \right| \leq \varepsilon |x-x_0|
\]

The expression on the right equals \( \varepsilon \cdot |x-x_0| \). (Don't worry! I'll go over this carefully in class!)

This implies that if \( x \) is within \( \delta \) of \( x_0 \), then:

\[
\left| \frac{1}{x-x_0} \int_{x_0}^{x} f(t)-f(x_0) \, dt \right| \leq \left| \frac{1}{x-x_0} \int_{x_0}^{x} \varepsilon \, dt \right| = \varepsilon \cdot \frac{|x-x_0|}{|x-x_0|} = \varepsilon
\]

This proves our theorem. Because, the expression on the far left is greater than or equal to the expression in our limit (where the \( \Delta \) is, near the top of the page). But as \( x \) goes to \( x_0 \), the thing on the left eventually becomes less than \( \varepsilon \), for any small \( \varepsilon \) we pick; i.e., it goes to zero as \( x \) goes to \( x_0 \). Therefore, the expression at the top of the page goes to zero, as \( x \) goes to \( x_0 \). Part I is proved.
Now that we have Part I, II. will be easy. Look back at page 15 to remember what II. says (unless you still remember it—but no one's memory could survive the proof of I!).

**Proof of II.** Let \( a \) be fixed, and define a function \( H(x) \) to be:

\[
H(x) = \int_{a}^{x} f(t) \, dt
\]

and let \( G(x) \) be any function that satisfies: \( G'(x) = f(x) \). We have to show that \( H(b) = G(b) - G(a) \). This is done by looking at the function:

\[
D(x) = H(x) - G(x)
\]

If we differentiate \( D(x) \), we get: \( D'(x) = H'(x) - G'(x) \). By Part I, \( H'(x) = f(x) \); and \( G'(x) = f(x) \). Thus, \( D'(x) = 0 \), for all \( x \). But the only functions which have zero slope everywhere are constant functions. Therefore, \( H(x) - G(x) = C \), where \( C \) is some constant. What is \( C \)? Well, if \( x = a \), then \( H(x) = 0 \). Since \( H(x) - G(x) = C \) for all \( x \), this must be true when \( x = a \):

\[
H(a) - G(a) = C
\]

\[
0 - G(a) = C
\]

\[
-G(a) = C
\]

Therefore, \( H(x) = G(x) - G(a) \). Now let \( x = b \), and we are done. Q.E.D.

**Example:** Let \( f(x) = x^n \), for \( n \neq -1 \). What is \( \int_{a}^{b} f(x) \, dx \)? If \( G(x) = x^{n+1}/(n+1) \), then \( G'(x) = f(x) \). By the Fundamental Theorem:

\[
\int_{a}^{b} f(x) \, dx = G(b) - G(a)
\]

\[
= \frac{1}{n+1} \left[ b^{n+1} - a^{n+1} \right]
\]

**Example:** Let \( f(x) = \sin 3x \). What is \( \int_{a}^{b} f(x) \, dx \)? Here our \( G(x) = -(1/3)\cos 3x \).

And so:

\[
\int_{a}^{b} f(x) \, dx = -(1/3)\cos 3b - \left[ -(1/3)\cos 3a \right]
\]

\[
= (1/3) \left[ \cos 3a - \cos 3b \right]
\]

**Homework:** Find these integrals.

1. \( \int_{a}^{b} (x^3 - 3x^2 + 2x - 5) \, dx \)
2. \( \int_{a}^{b} (2x)/(x^2 + 1) \, dx \)
3. \( \int_{a}^{b} \sec^2 x \, dx \)
4. \( \int_{a}^{b} \sec^2 5x \, dx \)
5. \( \int_{a}^{b} \sin x \, dx \)
6. \( \int_{a}^{b} \sqrt{x} \, dx \)
7. \( \int_{a}^{b} (f(x)^2) \, dx \)
1. Find these integrals.
   a. $\int_{0}^{\pi/4} \tan^2 x \, dx$
   b. $\int_{0}^{\pi/2} x \sin x^2 \, dx$
   c. $\int_{0}^{\infty} \frac{1}{(x^3+1)^2} \, dx$
   d. $\int_{0}^{1} (1 + x + x^2 + x^3 + \ldots + x^n) \, dx$
   e. $\int_{0}^{\pi} (\cos^2 x + \sin^2 x) \, dx$
   f. $\int_{0}^{\pi} \cos 2x \, dx$
   g. $\int_{0}^{\pi} \cos^2 x \, dx$  (Hint: What is $\cos 2x$? Then look at e. and f.)

2. Let $f(x) = \sqrt{1-x^2}$, for $x$ between $-1$ and $1$, inclusive. What is $\int_{-1}^{1} f(x) \, dx$?  (Hint: graph $f(x).$)

3. Extra credit problem. The natural logarithm of $x$ (for $x > 0$), denoted $\ln x$, is defined to be equal to:

   \[ \int_{1}^{x} \frac{dt}{t} \]

   There exists a function $y(x)$ which satisfies: $\ln (y(x)) = x$, for all $x$. Prove that $y'(x) = y(x)$, for all $x$.  (Hint: Use the chain rule to find $(\ln (y(x)))'$. )
Vectors and Kinematics

We are almost ready to begin doing physics. All we need now is a short look at vectors.

A vector is a thing that has a magnitude and a direction. Five feet down, twenty miles east, and a stone's throw thataway are all vectors. Quantities which have magnitude but no direction are called scalars. Twelve years, sixteen men on a dead man's chest, and 13 are scalars.

A vector is represented by an arrow, which points in the direction of the vector. The arrow's length is indicative of the magnitude of the vector. In writing, a vector is denoted like: \( \vec{a} \). The arrow over the 'a' shows that it is a vector.

Vectors can be added together and subtracted from one another. To see how this is done, look at the picture on the left. There, \( \vec{c} = \vec{a} + \vec{b} \), and \( \vec{d} = \vec{c} - \vec{a} \). We add vectors by joining the head of one to the other's tail, and then drawing their sum as a vector that points from the tail of one to the point of the other. We subtract them by first drawing the two vectors with their tails together, and then drawing a vector that goes from one's head to the other's. Why this is so (and how to know which way that vector points) will be clear in a moment.

Vectors can be multiplied by scalars. Above are the vectors \( \vec{x} \), \( 2\vec{x} \), \( 3\vec{x} \), and \( -2\vec{x} \).

Vector addition and scalar multiplication obey the following laws:

1. \( \vec{x} + \vec{y} = \vec{y} + \vec{x} \). (Commutativity)

2. \( \vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z} \). (Associativity)

3. \( a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y} \); where \( a \) is a scalar. (Distributivity)

4. \( (a+b)\vec{x} = a\vec{x} + b\vec{x} \); where \( a \) and \( b \) are scalars. (Distributivity again)

There is a special vector, called the zero vector, which has magnitude zero and no direction. It is denoted by \( \vec{0} \) or \( \vec{0} \). It has these properties:

5. \( \vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x} \). (Identity)

6. \( 0\vec{x} = \vec{0} \).

7. \( \vec{x} - \vec{x} = \vec{x} + (-\vec{x}) = \vec{0} \).

The magnitude or length of a vector is called its norm. The norm of a vector \( \vec{x} \) is denoted: \( ||\vec{x}|| \). The norm has these properties:

1. \( ||\vec{x}|| \geq 0 \).

2. \( ||\vec{x}|| = 0 \) if and only if \( \vec{x} = \vec{0} \).
3. \(|a\overline{a}| = |a||\overline{a}|\); \(a\) is any scalar.

4. \(|\overline{x} + \overline{y}| \leq |\overline{x}| + |\overline{y}|\). (Triangle Inequality)

The meaning of (4.) is shown below.

A vector with norm equal to one is called a unit vector. Three important unit vectors are: \(\hat{i}\), \(\hat{j}\), and \(\hat{k}\). The vector \(\hat{i}\) points in the positive direction on the x-axis; \(\hat{j}\) points in the positive direction on the y-axis; and \(\hat{k}\) points in the positive direction on the z-axis. See the diagram below.

To every point in space we assign a vector, called a position vector. This vector points from the origin to that point. In the diagram on the left, the position vector of the point \(P\) is \(\overrightarrow{OP}\).

I said that \(\hat{i}\), \(\hat{j}\), and \(\hat{k}\) were important. Why? They're important because any vector in space can be written in terms of them: as a sum of scalar multiples of them. The vector \(\overrightarrow{OP}\) in the diagram is equal to:

\[5\hat{i} + 2\hat{j} + 3\hat{k}\]

Such a sum is called a linear combination of \(\hat{i}\), \(\hat{j}\), and \(\hat{k}\). The vectors \(5\hat{i}, 2\hat{j}\), and \(3\hat{k}\) are called the components of \(\overrightarrow{OP}\). The vector \(5\hat{i}\) is called \(\overrightarrow{OP}\)'s x-component; \(2\hat{j}\) is its y-component; and \(3\hat{k}\) is its z-component. The components of any vector are unique. In other words, if for some vector \(\overrightarrow{OP}\):

\[p\hat{i} + q\hat{j} + r\hat{k} = \overrightarrow{OP} = d\hat{i} + e\hat{j} + f\hat{k}\]

then \(p = d\), \(q = e\), and \(r = f\). This is equivalent to saying that if:

\[\overrightarrow{i} + m\overrightarrow{j} + n\overrightarrow{k} = \overrightarrow{0}\]

then \(1 = m = n = 0\).

There are two ways in which we can multiply vectors together. One way is with the dot (or scalar) product. The dot product of \(\overrightarrow{a}\) and \(\overrightarrow{b}\) is denoted: \(\overrightarrow{a} \cdot \overrightarrow{b}\). It is a scalar, and it is equal to: \(\|\overrightarrow{a}\||\|\overrightarrow{b}\||\cos \theta\); where \(\theta\) is the angle between \(\overrightarrow{a}\) and \(\overrightarrow{b}\). See the diagram below.

The other way of multiplying vectors together is with the cross (or vector) product. The cross product of \(\overrightarrow{a}\) and \(\overrightarrow{b}\) is denoted: \(\overrightarrow{a} \times \overrightarrow{b}\). It's a vector. The direction of this vector is gotten in this (peculiar) way: Hold your right hand so that your fingers curl from \(\overrightarrow{a}\) to \(\overrightarrow{b}\). Then your thumb points in the direction of \(\overrightarrow{a} \times \overrightarrow{b}\).

The length of \(\overrightarrow{a} \times \overrightarrow{b}\) is: \(\|\overrightarrow{a}\||\|\overrightarrow{b}\||\sin \theta\), where \(\theta\) is the angle between \(\overrightarrow{a}\) and \(\overrightarrow{b}\). Again, you are referred to the diagram.

Notice: \(\overrightarrow{a} \times \overrightarrow{b}\) is perpendicular to both \(\overrightarrow{a}\) and \(\overrightarrow{b}\).
The dot and cross products have these properties:

1. \( \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \). (Commutativity)

2. \( \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \). (Distributivity)

3. \( \vec{a} \cdot (m\vec{b}) = (m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) \); \( m \) is any scalar.

4. \( \vec{a} \times \vec{b} = -(\vec{b} \times \vec{a}) \). (Anti-commutativity)

5. \( \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \). (Distributivity)

6. \( \vec{a} \times (m\vec{b}) = (m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b}) \); \( m \) is a scalar.

7. \((\vec{a} \cdot \vec{a}) = \|\vec{a}\|^2 \) (Why is this true?)

8. \( \vec{a} \times \vec{a} = \vec{0} \) (And why is this true?)

**Homework**

1. What is \( \vec{a} \cdot \vec{b} \) if \( \vec{a} \) and \( \vec{b} \) are perpendicular, and why?

2. Do these dot products.
   a. \((\vec{i} + \vec{j} + 2\vec{k}) \cdot (2\vec{i} - 3\vec{k})\)
   b. \((\vec{i} + 2\vec{k}) \cdot (2\vec{i} - 1\vec{j} + \vec{k})\)
   c. \((\vec{i} - \vec{j} + \vec{k}) \cdot (\vec{i} + \vec{j} - 2\vec{k})\)

3. Find the cross products of the vectors in Problem 2.

4. Prove that: \( \vec{x} \cdot \vec{y} = \frac{1}{2}(\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2) \).

**Kinematics**

Kinematics is the study of how one finds the position of a particle in space. Usually, this particle is moving (stationary particles are not very interesting). So, a problem in kinematics often consists in finding out what a particle's position vector is as a function of time. Such a position vector that is a function of time is denoted: \( \vec{r}(t) \). This vector has components which are also functions of time. We might write \( \vec{r}(t) \) as:

\[
\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}
\]

Usually, \( x(t) \), \( y(t) \), and \( z(t) \) have derivatives. These are denoted (in this special case, when they are functions of time): \( \dot{x}(t) \), \( \dot{y}(t) \), \( \dot{z}(t) \). They correspond to the components of the particle's velocity. Velocity, like position, is a vector. It is denoted: \( \dot{\vec{r}}(t) \). It equals:

\[
\dot{\vec{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \dot{z}(t)\hat{k}
\]

A particle's velocity is, roughly, how fast it's going and in what direction it's moving. A particle's speed is how fast it is moving, and it equals: \( \|\dot{\vec{r}}(t)\| \).
The third important vector quantity associated with a particle is its acceleration. We write it: \( \ddot{\mathbf{r}}(t) \). It equals:

\[
\ddot{\mathbf{r}}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}
\]

where \( \dot{x}(t) \) is the derivative of the derivative of \( x(t) = (\dot{x}(t))^2 \), etc.

**Homework.**

1. For each \( \ddot{\mathbf{r}}(t) \) below, find \( \dot{\mathbf{r}}(t) \) and \( \ddot{\mathbf{r}}(t) \).
   a. \( \ddot{\mathbf{r}}(t) = (\sin^2 t)\mathbf{i} + t^2 \mathbf{j} + (\tan t)\mathbf{k} \)
   b. \( \ddot{\mathbf{r}}(t) = (1/t)\mathbf{i} + (t/(t^2 + 1))\mathbf{j} + l\mathbf{k} \)
   c. \( \ddot{\mathbf{r}}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k} \)
   d. \( \ddot{\mathbf{r}}(t) = (\sin(t^2))\mathbf{i} + (\tan^2 t)\mathbf{j} + (\sin l t)\mathbf{k} \)
   e. \( \ddot{\mathbf{r}}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \)
   f. \( \ddot{\mathbf{r}}(t) = (\cos 10 t)\mathbf{i} + (\sin 10 t)\mathbf{j} \)

2. For \( \ddot{\mathbf{r}}(t) \) in (1.e) and (1.f), find:
   a. \( \ddot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t) \)
   b. \( \ddot{\mathbf{r}}(t) \cdot \ddot{\mathbf{r}}(t) \)

3. Write down an \( \ddot{\mathbf{r}}(t) \) for a particle that moves in a:
   a. straight line.
   b. circle.
   c. parabola.
The Kinematic Equation and Projectile Motion

The simplest kind of motion is motion in one dimension. In one-dimensional kinematics, we are interested in finding an \( x(t) \) for a particle's position on the \( x \)-axis as a function of time. The case we will look at is one in which the acceleration (\( a(t) \)) is constant. E.g., it equals \( a_0 \).

To find an \( x(t) \) for such a particle, it is sufficient to know \( x(0) \) and \( \dot{x}(0) \), which we will call \( x_0 \) and \( v_0 \). The Fundamental Theorem says that, for any \( t_0 \):

\[
\dot{x}(t) = \ddot{x}(t) = \int_{0}^{t} \dot{x}(t) \, dt
\]

Setting \( \ddot{x} = a \) gives us:

\[
\dot{x}(t_0) - \dot{x}(0) = \int_{0}^{t_0} \ddot{x}(t) \, dt = a t_0
\]

\[
(x)
\]

By the Fundamental Theorem, we also know:

\[
x(t_0) - x(0) = \int_{0}^{t_0} \dot{x}(t) \, dt
\]

When we substitute our result from \((x)\) we obtain:

\[
x(t_0) - x(0) = \int_{0}^{t_0} [v_0 + a t] \, dt = v_0 t_0 + \frac{1}{2} a t_0^2
\]

This is the kinematic equation for one-dimensional motion, with constant acceleration.

Near the surface of the Earth, bodies experience a nearly constant downward acceleration due to gravity; the magnitude of this acceleration is denoted by \( g \), and it equals (very nearly) 9.8 m/sec^2 = 32 ft/sec^2. Because of this, we can apply the kinematic equation in studying the motion of projectiles near the Earth's surface.

Problem: A cannon shoots a cannonball upward from ground level. The ball leaves the cannon at speed \( v_0 \), and air friction is negligible. How long is the ball in the air and how high does it get?

Here we let \( h(t) \) = ball's height as a function of time. We say that the cannon fires when \( t = 0 \), and that \( h(t) = 0 \) when the ball is at ground level.

Then we have:

\[
h(0) = 0
\]
\[
h(0) = v_0
\]
\[
h(t) = -g (\text{minus, because the acceleration is downward})
\]

Therefore:

\[
h(t) = 0 + v_0 t - \frac{1}{2} g t^2 = v_0 t - \frac{1}{2} g t^2
\]

To find the time in the air, we set \( h(t) = 0 \) and solve for \( t \). Then we discover that \( h(t) = 0 \) when \( t = 0 \) (when the cannon is fired) and when \( t = 2v_0/g \).

So the time in the air is \( 2v_0/g \).

To find the maximum height, we remember that \( h(t) \) will be at a maximum when \( \ddot{h}(t) = 0 \). We differentiate \( h(t) \) to get:

\[
\ddot{h}(t) = v_0 - gt
\]

This equals zero when \( t = v_0 / g \). Therefore:

\[
\text{max. height} = h(v_0 / g) = v_0 (v_0 / g - 1 \frac{g}{2v_0} (v_0 / g))^2
\]
Problem: A cannon makes an angle $\theta$ with the ground. Shooting from ground level, it fires a projectile with muzzle velocity $v_o$. Air friction is negligible. How far does the projectile go? How high does it get? How long is it in the air?

Our problem looks like this:

We want to find $x$, $y$, and $t_{\text{impact}}$. We do this by finding $\frac{dx}{dt} = x(t)i + y(t)j$. To make things easy, we pretend that the cannon is sitting at the origin.

First, let's notice that $\dot{x}(t) = 0$. This is because gravity acts downward, not horizontally, and because air friction is negligible. Therefore $\dot{x}(t)$ is a constant. That is, constant?

When the projectile leaves the cannon, its velocity has an $x$-component and a $y$-component. We can figure out what these are by looking at the diagram on the left. The $x$-component equals $v_o \cos \theta$ and the $y$-component equals $v_o \sin \theta$. (To save writing, from here on we denote $\sqrt{x}$ by $v_o$.) Thus $\dot{x}(0) = v_o \cos \theta = \dot{x}(t)$ (because $\dot{x}(t)$ is constant).

Since we've set up things to make $x(0) = 0$, we know that:

$$x(t) = 0 + (v_o \cos \theta) t$$

$$= (v_o \cos \theta) t$$

Our diagram shows that $\dot{y}(0) = v_o \sin \theta$. Gravity acts downward, making $\ddot{y}(t) = -g$. And $y(0) = 0$. The kinematic equation for $y(t)$ is:

$$y(t) = (v_o \sin \theta) t - \frac{1}{2} gt^2$$

Therefore: $\dot{y}(t) = (v \cos \theta) t + [v_o \sin \theta] (t - \frac{1}{2} gt^2)$.

The projectile is at ground level when $y(t) = 0$. This happens (solving for $t$ again) when $t = 0$ (when the cannon fires) and $t = (2v_o \sin \theta)/g$. The time in the air $T = (2v_o \sin \theta)/g$.

The number $h = x(T) = x$ ("when it hits the ground"), which equals:

$$x = (2v_o \sin \theta / g) = v_o \cos \theta \cdot (2v_o \sin \theta / g)$$

$$= (v_o^2 / g) \cdot 2 \cos \theta \cdot \sin \theta$$

$$= (v_o^2 / g) \sin 2\theta$$

The maximum height is $y(t)$'s maximum. This is reached when $\dot{y}(t) = v_o \sin \theta - gt$ equals zero. i.e., when $t = v_o \sin \theta / g$. Then, $y(t)$ equals (check it!):

$$h = \frac{1}{2} (v_o^2 / g) \sin^2 \theta$$
1. A rocket sled is put onto a track of length $L$. It accelerates down the track at a constant acceleration $a_r$, until it reaches the track's midpoint. Then its engine shuts down, and it slides for the rest of the way at constant velocity.

A jet-powered sled is put onto the same track. It accelerates for the whole length at a constant acceleration $a_j$.

The rocket and the jet take the same amount of time to travel the length of the track. Find $a_j/a_r$.

(The answer is $8/9$. I want to see how you would work it out.)
More Kinematics: the Homework

To solve the problem, we begin by re-stating it. The rocket sled accelerates for a time \( t_1 \), and slides without acceleration for a time \( t_2 \). The jet sled accelerates for a time \( t_1 + t_2 \).

The rocket sled starts from rest. The kinematic equation says that at time \( t_1 \), it has gone a distance:

\[
s = \frac{1}{2} a_R t_1^2
\]

The problem says that this distance \( s \) equals \( L/2 \).

Because the rocket sled starts from rest, its velocity at time \( t_1 \) is:

\[
v = a_R t_1
\]

This is the velocity at which it slides the rest of the way. The rest of the way is a distance of \( L/2 \), and it takes a time \( t_2 \) to slide it. Therefore:

\[
L/2 = vt_2
\]

\[
L/2 = (a_R t_1) t_2 = a_R t_1 t_2
\]

Look up at (*) and the note below it. We see that:

\[
\frac{1}{2} a_R t_1^2 = a_R t_1 t_2
\]

\[
\frac{1}{2} t_1 = t_2
\]

\[
t_1 = 2t_2
\]

Now, look at the jet sled. It accelerates for the whole length \( L \) for a time \( t_1 + t_2 \). The kinematic equation gives:

\[
L = \frac{1}{2} a_J (t_1 + t_2)^2
\]

\[
L/2 = \frac{1}{2} a_J (t_1 + t_2)^2
\]

The relation \( t_1 = 2t_2 \) turns this last equation into:

\[
L/2 = \frac{1}{2} a_J (2t_2 + t_2)^2
\]

\[
= \frac{1}{2} a_J (9t_2^2)
\]

Also, the equation (*) becomes:

\[
s = \frac{1}{2} a_R (2t_2)^2 = 2a_R t_2^2
\]

Since \( s = L/2 \), we get the equalities:

\[
2a_R t_2^2 = \frac{3}{2} a_J (9t_2^2)
\]

\[
(8/9) a_R = a_J
\]

\[
\left[ \frac{8}{9} = \frac{a_R}{a_J} \right]
\]

Voilà!
A Worked Problem

A cannon is at the bottom of a hill which slopes up at an angle of 45°. The cannon makes an angle θ with respect to the horizontal, and θ > 45°. It fires a projectile with muzzle velocity (speed) v₀. How far up the hillside does the projectile land? That is, in the diagram, what is L?

If we set the cannon at the origin, the \( \mathbf{r}(t) \) for the projectile is:

\[
(v_0 \cos \theta) t \mathbf{i} + \left[ (v_0 \sin \theta) t - \frac{1}{2} gt^2 \right] \mathbf{j}
\]

From our picture, we see that it will land on the hill when \( t = 0 \) and \( x(t) = y(t) \). This happens when:

\[
v_0 \cos \theta = v_0 \sin \theta - \frac{1}{2} gt
\]

\[
\frac{1}{2} gt = v_0 (\sin \theta - \cos \theta)
\]

\[
t = 2v_0 (\sin \theta - \cos \theta)/g
\]

For this \( t \), \( x(t) = (2v_0^2/g) (\sin \theta \cos \theta - \cos^2 \theta) \) and \( y(t) \) equals:

\[
(2v_0^2/g) (\sin^2 \theta - \sin \theta \cos \theta) - \frac{1}{2} g \left[ 2v_0 (\sin \theta - \cos \theta)/g \right]^2
\]

Now we have to figure out \( L = \sqrt{(x(t))^2 + (y(t))^2} \). That would be a mess, but fortunately we know that \( x(t) = y(t) \). Therefore:

\[
L = \sqrt{x(t)^2 + y(t)^2}
\]

\[
= \sqrt{2} \cdot |x(t)|
\]

\[
= \sqrt{2} \cdot (v_0^2/g) (\sin^2 \theta - 2 \cos^2 \theta)
\]

Homework

1. Two railroad trains are sitting on the x-axis; one at \( x = 0 \) and the other at \( x = L \) (they're very short trains). They are initially at rest. At time \( t = 0 \), they begin to accelerate toward each other at constant rates \( a_1 \) for the train that begins at \( x = 0 \) and \( a_2 \) for the other train. They collide at the point \( x = \alpha L \), where \( 0 < \alpha < 1 \). What is \( a_1/a_2 \)?

2. A ball is dropped from a height \( h \) above a cannon which is pointing straight up. Just as the ball is dropped, the cannon fires a short person. If the short person catches the ball at height \( h/2 \), what was his muzzle speed? (And what was he doing with a fast muzzle?)
TEST #5 (Assume that gravity is acting in all three problems)

1. Two cannons sitting at the origin fire projectiles simultaneously. The muzzle speeds of the projectiles are equal. If one cannon makes an angle \( \theta_1 \) with respect to the x-axis, and the other cannon makes an angle \( \theta_2 \), where \( \theta_1 \neq \theta_2 \), what is the slope of the line segment which connects the two projectiles in flight?

2. Evil Knievel rides his motorcycle up a ramp (of length \( L \) and inclination \( \theta \)) at a constant acceleration \( a \). Then he jumps across a gap of length \( R \) to come down onto an identical ramp. Find \( a \), in terms of \( R \), \( L \), \( g \), etc. (Hey, is \( a \) nm \( \frac{R}{t} \) \( \text{the} \) \( \text{bottom} \) \( \text{of the ramp} \).) \( \text{——-} = \text{Evil's} \ P \text{m} \text{R} \)

3. Peter Cottontail is hopping along the x-axis with a constant horizontal velocity of \( v_x \). He comes to a rabbit-trap of length \( L \). At the edge of the trap, he gives himself an upward velocity of magnitude \( v_y \), while keeping his horizontal velocity unchanged. He just barely clears the trap. What is \( v_y \), in terms of \( v_x \), \( L \), and \( g \)?

WHERE RABBITS FEAR TO GO!
Forces

If a body with mass is undergoing acceleration, we say that a force is acting on it. The force is a vector, equal to the body's mass times its acceleration. In symbols:

\[ \mathbf{F} = m \mathbf{a} \]

Since force is a vector, we can add forces together. If several forces—\( F_1, F_2, F_3, \ldots, F_k \)—are acting on a body, then the total force, \( \mathbf{F} \), acting on it is just their vector sum:

\[ \mathbf{F} = \sum_{k=1}^{n} F_k \]

If the total force on a body is 0, then the body is not being accelerated, and vice versa. This is Newton's first law: that a body not being acted upon by a force will either remain at rest or move in a straight line.

Near the surface of the Earth, a body of mass \( m \) experiences a downward force of magnitude \( mg \). The quantity \( mg \) is called the body's weight.

We usually think of forces as pushing bodies, but they can also pull them: through a string, for instance. When something is being pulled by a string, the force that is acting through that string is called the string's tension. Its magnitude is equal at both ends of the string.

In the picture below, the masses \( m_1, m_2, \) and \( m_3 \) are connected by massless, unstretchable strings. The pulley is massless, and \( m_2 \) and \( m_3 \) are sliding without friction. If \( m_1 \) is being pulled down by gravity, what is the magnitude of the masses' acceleration? What is the tension in the string connecting \( m_2 \) and \( m_3 \)?

![Diagram of three masses connected by strings](image)

The only force acting on the masses is the force of gravity acting on \( m_1 \). The magnitude of this is: \( m_1 g \). Therefore, letting \( F \) be the magnitude of the total force:

\[ F = m_1 g \]

But this force is acting through the strings, on all three masses, causing them to accelerate with some acceleration \( a \). Hence:

\[ F = (m_1 + m_2 + m_3) a \]

Which means that:

\[ m_1 g = (m_1 + m_2 + m_3) a \]

\[ a = m_1 g / (m_1 + m_2 + m_3) \]

The tension in the string between \( m_2 \) and \( m_3 \)—call it \( T \)—is the force which is acting directly on \( m_3 \), causing it to accelerate at the rate \( a \). Therefore:

\[ T = m_3 a \]

\[ = m_3 m_1 g / (m_1 + m_2 + m_3) \]
If a body of mass \( m \) is hanging by a cord, what is the tension in the cord? Since the mass isn't moving, the total force on the mass must be 0. The only forces acting on the mass are that of gravity (pulling down) and the cord's tension (pulling up). These must cancel; their magnitudes must be equal. The magnitude of the gravitational force is \( mg \). The tension is, therefore, \( mg \). See the picture.

Suppose that the hook to which the cord is tied begins accelerating upward at a rate \( a \). What is the tension in the cord then? The upward force, \( F \), is equal to the body's mass times its acceleration:

\[ F = ma \]

But this force is the sum of two forces: the cord's tension (upward) and gravity (downward). Therefore:

\[ F = T - mg \]

Or:

\[ ma = T - mg \]

\[ T = m(a + g) \]

If a body is sitting on a ramp, we can write the gravitational force acting on it as the sum of two components. One, called the normal force, acts perpendicularly to the ramp's surface. The other, called the transverse force, acts parallel to the ramp's surface. If the ramp is tilted at an angle \( \theta \), and the body's mass is \( m \), then the magnitude of the normal force is: \( mg \cos \theta \). And the magnitude of the transverse force is: \( mg \sin \theta \). To see why this is so, look at the diagram on the left.

Suppose that a body of mass \( m \) slides without friction down from the top of a ramp with base length \( L \) and inclination \( \theta \). How long does \( m \) (the body) take to reach the bottom of the ramp? See the picture above.

The transverse force acting on \( m \) (which is what will make it slide) has magnitude \( mg \sin \theta \). This must equal \( m \)'s mass \( (=m) \) times its transverse acceleration, which we will call \( a \). In symbols:

\[ ma = mg \sin \theta \]

\[ a = g \sin \theta \]

The length which \( m \) slides equals \( L / \cos \theta \). If \( t \) is the time \( m \) is sliding, then:

\[ \frac{L}{\cos \theta} = \frac{1}{2} at^2 \]

\[ t = \sqrt{\frac{2L}{(a \cos \theta)}} = \sqrt{\frac{2L}{(g \sin \theta \cos \theta)}} \]

How fast is \( m \) going when it reaches the bottom?
1. In the diagram below, the masses of the bodies are as marked. The pulley is massless and frictionless, and everything rolls without friction. The force $F$ is such that $m_3$ and $m_2$ do not move relative to one another ($m_2$ is being pulled down by gravity). What is $F$'s magnitude? 
(Answer: $(m_2 g/m_1)m(m_1 + m_2 + m_3)$. Show me how to get it.)

![Diagram of masses and pulley](image)

2. The $\mathbf{f}(t)$ for a body of mass $m$ is: $\cos(kt)\mathbf{i} + \sin(kt)\mathbf{j}$, where $k$ is a constant. What is $F(t) = \text{force acting on } m$?

3. Another body, of mass $m'$, has an $\mathbf{f}(t)$ equal to: $A\cos(\pi t)\mathbf{i} + A\sin(\pi t)\mathbf{j}$, where $A$ and $\pi$ are constants. What is $F'(t) = \text{force acting on } m'$? If $\sqrt{F'(t)^2} = \sqrt{F(t)^2}$, what is $m/m'$?

4. In the diagram below, what is $m_2$'s upward acceleration, in terms of $m_1$, $m_2$, $\theta$, etc. (The pulley is massless and frictionless, the cord is massless, and $m_1$ slides without friction.)

![Diagram of forces and masses](image)

5. In the picture below, the body of mass $m$ slides without friction. The ramp has inclination $\theta$ and base length $L$. A constant force $F$ is pushing $m$ up the ramp. If $m$ starts from the bottom of the ramp at rest, what must $F$'s magnitude be, in order that $m$'s speed at the top be $v$?

![Diagram of ramp and force](image)
The Wall Problem

For those of you who have forgotten (deliberately, no doubt) the problem we were looking at yesterday, it was the following: A cannon is a distance $L$ from a wall of height $h$. What is the smallest muzzle velocity $v_0$ which the cannon can have, and still get a projectile over the wall? The problem is presented in the picture below:

![Diagram](image)

The minimum $v_0$ will be that one for which the projectile just grazes the top of the wall. If the inclination of the cannon were fixed at some angle $\theta$, then we could—without too much trouble—find the $v_0$ for that $\theta$ which would send the shell just over the wall. But we are looking for the minimum $v_0$ for all angles $\theta$: the smallest of the smallest, so to speak. Therefore, when we have found what the minimum $v_0$ is (as a function of $\theta$), we will have to differentiate that function and find for what $\theta$ its derivative is zero. The $v_0$ which corresponds to that angle will be the answer.

That is the idea. But actually, it will turn out that $v_0^2$—as a function of $\theta$—is a much less messy function than $v_0$. So we'll look at $v_0^2$ and find for what $\theta$ it is a minimum. This is all right, because if we know the minimum value of $v_0^2$, we also know the minimum value of $v_0$.

Now let us find $v_0^2$ for a fixed $\theta$. If the cannon fires a shell at speed $v_0$ and angle $\theta$, the $\vec{r}(t)$ for the shell is:

$$\vec{r}(t) = (v_0 \cos \theta) t \hat{i} + (v_0 \sin \theta - \frac{1}{2}gt^2) \hat{j}$$

At some time the shell must be at the top of the wall. The position vector of that point is: $L \hat{i} + h \hat{j}$. Therefore, for some $t$:

$$\vec{r}(t) = L \hat{i} + h \hat{j}$$

$$(v_0 \cos \theta) t \hat{i} + (v_0 \sin \theta - \frac{1}{2}gt^2) \hat{j} = L \hat{i} + h \hat{j}$$

Since the $x$- and $y$-components of vectors are unique (see p. 19), this can only happen when:

$$(v_0 \cos \theta) t = L$$

$$(v_0 \sin \theta - \frac{1}{2}gt^2) = h$$

The first equation yields a value for $t$: $t = L/(v_0 \cos \theta)$. When we substitute this $t$ into the second equation, we get:

$$v_0 \sin \theta \left[ L/(v_0 \cos \theta) \right] - \frac{1}{2}g \left[ L/(v_0 \cos \theta) \right]^2 = h$$

Fortunately, this can be simplified. At first, to:

$$L \tan \theta - \frac{1}{2}gL^2/(v_0 \cos \theta) = h$$
This equation transforms into:

\[ L \tan \theta - h = \frac{1}{2} g L^2 / (v_0^2 \cos^2 \theta) \]

\[ L \tan \theta - h = \left( \frac{1}{2} g L^2 / \cos^2 \theta \right) \times (1/v_0^2) \]

The last equation implies that:

\[ \frac{1}{v_0^2} = \left( L \tan \theta - h \right) \times 2 \cos^2 \theta / (gL^2) \]

\[ = \frac{2 \sin \theta \cos \theta}{gL} - \frac{2 \cos^2 \theta}{gL^2} \]

Therefore, when we invert both sides of this equation, we get:

\[ v_0^2 = \left( \frac{2 \sin \theta}{gL} - \frac{2 \cos^2 \theta}{gL^2} \right)^{-1} \tag{*} \]

The first half of our task is finished. To do the second half, we take the derivative (with respect to \( \theta \)) of the expression on the right, and find when it equals zero. The derivative of that thing is:

\[ - \left( \frac{2 \cos \theta}{gL} + \frac{2 \sin 2 \theta}{gL^2} \right) \left/ \left( \frac{2 \sin \theta}{gL} - \frac{2 \cos^2 \theta}{gL^2} \right) \right.^2 \]

That will be zero when its numerator is zero, which will be when:

\[ 2 \cos 2 \theta / (gL) = -2 \sin 2 \theta / (gL^2) \]

\[ (L/h) \cos 2 \theta = -\sin 2 \theta \]

\[ -L/h \tan 2 \theta \]

Since pointing the cannon at this angle sends the shell over the wall, \( \theta \) must be less than or equal to 90°. Therefore, 2\( \theta \) is less than or equal to 180°. This means that \( \sin 2 \theta \) is non-negative (draw a unit circle, if you don’t see why). So \( \cos 2 \theta \) is negative.

We have these relations between \( \sin 2 \theta \) and \( \cos 2 \theta \):

\[ L^2 / h^2 = \tan^2 2 \theta \]

\[ = \sin^2 2 \theta / \cos^2 2 \theta \]

\[ \cos^2 2 \theta \times L^2 / h^2 = \sin^2 2 \theta = 1 - \cos^2 2 \theta \]

These relations (and the fact that \( \cos 2 \theta \) is negative) imply, after a little algebra:

\[ \cos 2 \theta = -h / \sqrt{L^2 + h^2} \]

\[ \sin 2 \theta = L / \sqrt{L^2 + h^2} \]

And now the second half of our task is almost done. We use the fact that:

\[ \cos 2 \theta = \cos^2 \theta - \sin^2 \theta \]

\[ = 2 \cos^2 \theta - 1 \]
$$\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta)$$

Now we look back on page 29 at equation (#). We substitute in our values for $\sin 2\theta$ and $\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 - h/\sqrt{L^2 + h^2})$. We get:

$$V_o^2 = \left[ \frac{L\sqrt{L^2 + h^2}}{gL} - \frac{2h\sqrt{L^2 + h^2}}{gL^2} \right]^{-1}$$

$$= \left[ \frac{L}{\sqrt{L^2 + h^2}} - \frac{h}{gL^2} \sqrt{L^2 + h^2} \right]^{-1}$$

$$= \left[ \frac{L^2 + h^2 - h\sqrt{L^2 + h^2}}{gL^2 \sqrt{L^2 + h^2}} \right]^{-1}$$

$$= \left[ \frac{\sqrt{L^2 + h^2}}{gL^2} \frac{h}{\sqrt{L^2 + h^2}} \right]^{-1}$$

$$= L^2 g/(\sqrt{L^2 + h^2} - h)$$

And we are done, done, done, done, done, done, done, done . . .

Note: this can be simplified further to:

$$V_o^2 = g \left( \sqrt{L^2 + h^2} + h \right)$$
1. A penguin is flying south for the winter. He is flying at a constant height $h$ over the Atlantic Ocean, and at a constant velocity $v = 100$ feet per second, due south. As he is passing by the Isles of Langerhans, he looks behind him and sees an APM (Anti-Penguin Missile). Immediately, he lets his Samsonite suitcase fall from his bill, and begins evasive action. The suitcase falls into the sea at a distance $2h$ south of where the penguin let go of it. Find $h$ (it'll be a number).

2. A particle on the $x$-axis is initially at rest at the origin. At time $t = 0$, it begins accelerating in the positive direction on the $x$-axis at a constant rate $a_1$. It does this until it reaches the point $x = L$. Then it begins to experience a negative acceleration of magnitude $a_2$. At time $t = T$, it passes through the origin. What is $T$?

3. Three roads form a right triangle, as below, with side length $AB$ equal to $L$. Simultaneously, two cars start from points $A$ and $B$ and move toward $C$ at constant velocities $v_o$ and $v$ (see the diagram). They meet at $C$. What is $\tan \theta$? What is their travel time?

4. Darth Vader is playing basketball. He throws the ball at a basket that is a distance $L$ away and a height $h$ above his hands. He throws it at an angle of $30^\circ$ and speed $v_o$. The ball goes through the basket with no rebound. Find $v_o$. Assume that the Force of gravity is as it is on Earth.
More Forces: Answers to Homework

1. The force being applied to \( m_3 \) is going to cause the system of masses to accelerate with some acceleration \( a \). Knowing \( F \) will give us \( F \), since \( F \) = horizontal force on system = (mass of system) \( \times a \) = \( (m_1 + m_2 + m_3) \times a \). We must find \( a \).

Look at \( m_1 \), in the diagram. We are given that it isn't sliding on \( m_3 \); hence, its acceleration must be the same as \( m_1 \)'s. This means that a force must be acting on \( m_1 \), equal to \( m_1a \).

The only horizontal force that is acting directly on \( m_1 \) is from the tension in the cord that connects it to \( m_3 \). The magnitude of this force is equal to the tension, \( T \). The preceding paragraph gives us:

\[
T = m_1a
\]

The tension in the cord also acts on \( m_2 \). It must be equal in magnitude to the force of gravity that is trying to pull \( m_2 \) down (otherwise, \( m_2 \) would slide). Therefore:

\[
T = m_2g
\]

Which, with the last equation, implies:

\[
m_1a = m_2g
\]

\[
a = \frac{m_2g}{m_1}
\]

Therefore: \( F = (m_2g/m_1) \times (m_1 + m_2 + m_3) \)

2. Easy. Force = mass \( \times \) acceleration = \( m_2 \ddot{r}(t) \), which equals:

\[
= -k_2m_2m_1 \cos \theta t^2 - k_2m_2m_1 \sin \theta t
\]

\[
= -k_2m_2 \ddot{r}(t)
\]

3. Easy again. The force is:

\[
= -h^2m_1A \cos(\theta t)^2 - h^2m_1A \sin(\theta t)^2
\]

\[
= -h^2m_1 \ddot{r}(t)
\]

The norm of the force in (2.) is \( k_2m_2 \). The norm of the above force is \( h^2m_1A \). If they are equal, then:

\[
m_1A = h^2A / k^2
\]

4. To do this problem, let's first define a direction around the pulley to be positive. See the diagram below. The direction of
the arrow will be this direction. So, if \( m_2 \) goes up, its acceleration is positive; if \( m_1 \) goes up, its acceleration is negative.

The forces acting on \( m_2 \) are a positive tension, \( T \) (pulling up), and a negative gravitational force, \( m_2g \) (pulling down). Since these are all the forces acting on \( m_2 \), their difference must be equal to \( m_2 \) times the mass's acceleration \( = m_2a \). Thus:

\[
m_2a = T - m_2g
\]

The mass \( m_1 \) must be experiencing an equal acceleration. The forces acting on it are a negative (upward) tension \( T \), and a positive (downward) transverse gravitational force \( = m_1gsin\theta \). As for \( m_2 \), their difference must equal \( m_1a \):

\[
m_1a = m_1gsin\theta - T
\]

If we add this equation to the above equation, we get:

\[
(m_1 + m_2)a = m_1gsin\theta - m_2g
\]

\[
a = (m_1gsin\theta - m_2g)/(m_1 + m_2)
\]

\[
= g(m_1sin\theta - m_2)/(m_1 + m_2)
\]

5. The total force acting on \( m \) is the result of the force \( F \), which is pushing \( m \) up hill, and the force of gravity -- \( mgsin\theta \) -- which is pulling \( m \) downhill. If \( m \)'s uphill acceleration is \( a \), then:

\[
ma = F - mgsin\theta
\]

\[
(*) \quad a = (F - mgsin\theta)/m
\]

The distance over which \( m \) accelerates is \( L/cos\theta \). Therefore, the speed that it will have at the top of the ramp \( = v_0 = \sqrt{2L/cos\theta}a \).

Therefore:

\[
v_0^2 = 2La/cos\theta
\]

\[
\frac{1}{2}v_0^2 = \frac{1}{2}v_0^2cos\theta/L = a
\]

When we combine this with equation (\(*\)), we get:

\[
\frac{1}{2}v_0^2cos\theta/L = (F - mgsin\theta)/m
\]

\[
\frac{1}{2}mv_0^2cos\theta/L = F - mgsin\theta
\]

\[
mgsin\theta + \frac{1}{2}mv_0^2cos\theta/L = F
\]

\[
F = m(gsin\theta + \frac{1}{2}v_0^2cos\theta/L)
\]
1. On the planet Tatooine, the acceleration of gravity is 10 m/sec². R2D2 and C-3PO are landing on Tatooine in their space pod. The mass of the pod is 1000 kg; R2D2 and C-3PO each have a mass of 100 kg. When they are 10 km above the planet's surface, their velocity is $v = 1000$ m/sec, straight down. At this time, the jets on the pod come on, creating an upward thrust $F$, which remains constant during the pod's descent. When they touch down, their speed is zero. How big is $F$ in newtons? Ignore air friction and the variation of $g$ with altitude; also, assume that the mass of the fuel used in the descent is negligible.

2. A particle of mass $m$ is sitting at the point $x = 0$. At time $t = 0$, a constant force $F$ begins to push it along the positive direction of the $x$-axis. At time $t = T$, when the particle is at $x = L$, the force stops, and the particle then moves with a constant speed $v$. Prove that:

$$FL = \frac{1}{2}mv^2$$

$$FT = mv$$

3. In the diagram below, the pulleys are massless and frictionless, and the cords are massless. Also, $m_1 = m_2$. What is $m_1$'s acceleration (don't forget the direction) in terms of $g$? With how much force are the masses, through the cords, pulling down on the bar $B$?

4. In the system below, the pulleys are massless, etc., etc. The mass of $m_1$ is 2 kg. If the system is in equilibrium, what are $m_2$ and $m_3$?
Springs and Friction

A spring exerts a force on masses attached to it. This force is proportional to the spring's displacement from some equilibrium position, $x_0$, and is in the opposite direction to the displacement. See the diagram.

$$F = -k(x - x_0)$$

This property of springs is expressed in the equation: $F = -k(x - x_0)$, or just $F = -kx$, if $x$ is the displacement from $x_0$. The constant $k$ is called the spring constant, and the equation is Hooke's Law.

Hooke's Law is not exact, but an approximation. The approximation is good when the displacements of the spring are small—so good that, for our purposes, we can use it as if it were exact.

Problem: A mass $m$ is hanging from a spring with spring constant $k$. The mass is not moving. What is $x$ = spring's displacement from the equilibrium position? See the diagram.

The total force on $m$ is zero. The upward force equals $kx$. The downward force, $m$'s weight, is $mg$. Then:

$$kx - mg = 0$$

$$x = \frac{mg}{k}$$

Note that if $m$ were zero, the spring would not be stretched at all. Also, if the spring were infinitely stiff—if $k$ equaled infinity—it would not be stretched. Likewise, if there were no gravity.

Springs are something like cords: they pull as hard from one end as from the other. We use this fact in the following problem: If two springs with constants $k_1$ and $k_2$ are connected in series, what is the spring constant $K$ of the new super-spring? See the picture.

We are asking: If the super-spring is stretched by an amount $x$, with how much force does the spring pull back? Or: How much force is needed to keep the spring stretched by an amount $x$?

Let us suppose that we have a force $F$, being applied at the point $P$, where $F$ is such that the spring is not moving, and the spring is stretched by an amount $x_0$. In so stretching the super-spring, we have stretched the two smaller springs by amounts $x_1$ and $x_2$. This is all the stretching that's going on, so: $x = x_1 + x_2$.

Since the springs aren't moving, the total force at $P$ must be zero. The only forces acting directly at $P$ are $F$ and the force from spring number two. This latter force is $-k_2x_2$. Hence:

$$F - k_2x_2 = 0$$

$$F = k_2x_2$$

$$\frac{F}{k_2} = x_2$$
The sum of the forces at \( \frac{1}{2} \) where the springs join—must also be zero. Spring number two is pulling to the right with a force \( k_2 x_2 \). Spring number one is pulling to the left, with a force of magnitude \( k_1 x_1 \). As before, we have:

\[
k_2 x_2 = k_1 x_1
\]

But we also know that \( F = k_2 x_2 \). Therefore:

\[
F = k_1 x_1
\]

\[
\frac{F}{k_1} = x_1
\]

\[
\frac{F}{k_1} + \frac{F}{k_2} = (x_1 + x_2) = x
\]

By definition, \( kx = F \). So we get:

\[
K(\frac{F}{k_1} + \frac{F}{k_2}) = F
\]

\[
K(\frac{1}{k_1} + \frac{1}{k_2}) = 1
\]

\[
K = 1 / (\frac{1}{k_1} + \frac{1}{k_2}) = k_1 k_2 / (k_1 + k_2)
\]

HOMEWORK

1. Prove, by mathematical induction, that if \( n \) springs, with constants \( k_1, k_2, k_3, \ldots, k_n \) are connected in series, then \( K \) the spring constant of this super-spring satisfies:

\[
\frac{1}{K} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}
\]

Another source of forces is friction. When two surfaces rub against each other, microscopic irregularities on them create a force opposed to the direction of the rubbing (the surfaces' motion), and which is proportional to the normal force—the force that presses them together. This relation is expressed by the equation:

\[
\tau = \mu N
\]

\( \tau \) is the frictional force, \( N \) is the magnitude of the normal force, and \( \mu \) is called the coefficient of friction. The value of \( \mu \) is a function of the types of material that are being rubbed together. The \( \mu \) for Teflon on Teflon is much lower than the \( \mu \) for sandpaper on sandpaper.

Problem: A mass \( m \) is sitting on a flat surface, under normal gravity. The coefficient of friction between it and the surface is \( \mu \). What force \( F \) must be applied to it to give it an acceleration of \( a? \)

The normal force here is just \( m \)‘s weight: \( mg \). Therefore, the frictional force \( F = \mu mg \).

The total force on \( m \) is \( ma \). This will be the result of \( F \)‘s pushing to the right and \( F \)‘s pushing to the left. Therefore:
ma = \vec{F} - \vec{\tau} \\
= \vec{F} - \mu mg \\
F \leq m(a + \mu g)

If \( m \) is sliding without friction (\( \mu = 0 \)), then \( F = ma \), as we would expect.

Problem: A mass \( m \) is sitting on a ramp of inclination \( \Theta \). How big does the \( \mu \) between the mass and the ramp have to be, so that \( m \) won't slide down the ramp?

We want to find the \( \mu \) for which the frictive force exactly balances the transverse gravitational force. This force has magnitude \( mgsin\Theta \).

The normal force, we saw earlier, has magnitude \( mgcos\Theta \). So here,

\( \tau = \mu mgcos\Theta \)

Since these forces must balance, we have the equation:

\[ mgsin\Theta = \mu mgcos\Theta \]

\[ \mu = \frac{sin\Theta}{cos\Theta} = tan\Theta \]

Does this make sense for limiting cases (\( \Theta = 0 \) and \( \Theta = \pi/2 \))?

**Homework**

2. At time \( t = 0 \), a mass \( m \) is sliding with speed \( v \) across a flat surface. The only horizontal force acting on \( m \) is due to friction with the surface, for which the coefficient of friction is \( \mu \). This happens under normal gravity.

At what time \( t = T \) does \( m \) come to a full stop? And how far is it then from its position at \( t = 0 \)?

3. The masses \( m_1 \), \( m_2 \), and \( m_3 \) are connected by cords, as below. The pulley is frictionless and massless, and the cords are massless. The coefficient of friction between \( m_2 \) and the surface is \( \mu_2 \); that for \( m_3 \) is \( \mu_3 \). Assuming that the \( \mu \)'s are small enough to let \( m_1 \) fall, what is \( m_1 \)'s downward acceleration?

(Extra credit: Prove that \( m_1 \) will fall if and only if: \( m_1 > \frac{\mu_2}{\mu_3} m_2 + \frac{1}{\mu_3} m_3 \))

4. Do number five on the last homework set, but now let there be a coefficient of friction \( \mu > 0 \) between \( m \) and the ramp.
TEST #8 On all problems, ignore air friction and assume that the springs are massless.

1. Star pilot William Brent is landing his ship, the Decade Dodo, on the planet Tatooine (where \( g = 10 \, \text{m/sec}^2 \)). When the Dodo touches down, her landing gear break, and she skids on her belly, with an initial horizontal speed of 1000 m/sec. The coefficient of friction between her belly and the ground is a constant, \( \mu \). She is heading straight for a cliff, 12.5 km away. Captain Brent is relaxed. "Plenty of time to stop," he says. Unfortunately, when the Dodo reaches the cliff, her speed has only been reduced by half, and she goes over. Find \( \mu \).

\[ \begin{align*}
\text{1000 m/sec} \\
12.5 \text{ km}
\end{align*} \]

2. A spring has a constant \( k \). If I cut the spring in half, what is the constant of the half-spring?

\[ \text{Constant} = 2 \]

3. In the diagram below, the coefficient of friction between the mass \( m \) and the ramp is \( \mu \). The spring has constant \( k \). How far downhill from the spring's equilibrium position can the mass be set before the spring starts pulling it back uphill?

4. In the system below, the spring constants and masses are as marked. The system is in equilibrium. The springs have been stretched by amounts \( x_1 \) and \( x_2 \). What are \( x_1 \) and \( x_2 \)?
Momentum and Energy

To a particle of mass \( m \) we associate a vector quantity, called the particle's momentum, which we denote by: \( \vec{p} \); and which is equal to the particle's mass times its velocity. In symbols:

\[
\vec{p} = m \vec{v}
\]

If we have a system of particles with masses \( m_1, m_2, m_3, \ldots, m_n \) and respective velocities \( v_1, v_2, v_3, \ldots,v_n \), then the total momentum of the system is equal to:

\[
\vec{P} = \sum_{i=1}^{n} m_i \vec{v}_i
\]

I.e., the sum of the individual momenta.

The vector \( \vec{P} \) is always defined with respect to some reference frame—"point of view." For example, relative to me, my total momentum is 0: I am not moving, relative to myself. But if somebody is going by (or at) me in an automobile, I have a non-zero momentum relative to him. This is an important fact.

If no outside forces are acting on a system, then the total momentum \( \vec{P} \) does not vary with time. In symbols:

\[
\frac{d\vec{P}}{dt} = 0
\]

This is the Law of the Conservation of Momentum.

But what exactly is an outside force? An outside force is one which is being exerted by a particle (or body) that is outside the system. In the diagram below, the black bodies are in the "system." The short person is outside the system. His interaction (through the cord) with the dragon is changing the system's \( \vec{P} \). However, the interaction between the squares is not affecting it. The Law says that no such interactions, no matter what their nature, can act to change the system's total momentum.

But the Law, as spectacular as it may look, is really just a dressed-up definition. Because the total force \( \vec{F} \) that is acting on a system of particles is defined as:

\[
\vec{F} = \frac{d\vec{P}}{dt}
\]

So of course, when \( \vec{P} = 0 \), so does \( \frac{d\vec{P}}{dt} \). This definition of force is the one that Newton used. It is really more correct (and useful) than just \( \vec{F} = \vec{a} \). And the old formula is hidden in this one. For, suppose that we have one particle, of mass \( m \). This new formula says that:

\[
\vec{F} = \frac{d\vec{P}}{dt} = \frac{d(m \vec{v})}{dt} = m \frac{d\vec{v}}{dt} = m \vec{a}
\]
\[-\frac{d(m\vec{v})}{dt} = m(d\vec{v}/dt) + \vec{v}(dm/dt)\]

by the product rule for differentiation. Now, \(d\vec{v}/dt = \vec{a}\) is just the acceleration. And, if the particle's mass isn't changing, then \(dm/dt = 0\). Therefore, for one particle with constant mass \(m\):

\[\vec{F} = ma\]

But this formula only has meaning for one-body systems, or systems which we can treat as one body (as when several masses are connected by cords). The other formula works for a whole system of masses, which might be interacting in all kinds of crazy ways.

**Problem:** Two masses, \(m_1\) and \(m_2\), are sitting on a frictionless plane. They are stuck together, with an explosive charge between them. The charge explodes, and \(m_1\) goes sliding with a velocity \(\vec{v}_1\). What is \(m_2\)'s velocity?

Before the explosion, the momentum of this system is 0. The conservation law says that it must be 0 afterwards as well. Therefore:

\[m_1\vec{v}_1 + m_2\vec{v}_2 = 0\]

\[\vec{v}_2 = -(m_1/m_2)\vec{v}_1\]

This is Newton's "action-reaction" principle.

**Problem:** A mass \(m_1\) is sliding on a frictionless plane with velocity \(\vec{v}_1\). It hits a stationary mass \(m_2\) and sticks to it. What is the velocity \(\vec{v}\) of the pair after the collision?

The total momentum of the system before the collision is:

\[m_1\vec{v}_1 + m_2\vec{v} = m_1\vec{v}_1\]

The momentum after the collision is:

\[(m_1 + m_2)\vec{v}\]

The two momenta must be equal. Therefore:

\[(m_1 + m_2)\vec{v} = m_1\vec{v}_1\]

\[\vec{v} = m_1\vec{v}_1 / (m_1 + m_2)\]

(Check: Does this answer make sense for limiting cases?)

Another quantity associated with moving particles is kinetic energy. The kinetic energy of a particle with mass \(m\) and speed \(\vec{v}\) is:

\[KE = \frac{1}{2}mv^2\]

The kinetic energy is a measure of a particle's ability to do work. We will look at work and potential energy later on. For now, we are interested in how kinetic energy and momentum are related to particle-particle collisions.
Just as we could define a total momentum $\vec{p}$ for a system, we can also define a total internal kinetic energy for a system of masses $m_1, m_2, \ldots, m_n$, with speeds $v_1, v_2, \ldots, v_n$:

$$KE = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$$

This kinetic energy is called internal because it is often, in a sense, trapped within the system. For instance, the internal kinetic energy of a stationary baseball at room temperature is rather high: it is the sum of the $\frac{1}{2} m_i v_i^2$'s of all the baseball's atoms. Yet the kinetic energy of the baseball itself is zero.

We are interested in looking at systems of two particles; in which, at some time, they collide. If two particles collide so that their total kinetic energy is the same before and after the collision, then that collision is called elastic. If the kinetic energy changes, then the collision is inelastic.

**Problem:** As the collision on page 37 elastic or inelastic? If we let $v_i$ denote $|\vec{v}_i|$, to save writing (and eyestrain), then we see that the total KE before the collision is:

$$KE = \frac{1}{2} m_i v_i^2$$

The kinetic energy after the collision is:

$$KE' = \frac{1}{2} (m_1 + m_2) \left| \frac{m_1 v_1}{m_1 + m_2} \right|^2$$

$$= \frac{1}{2} m_2 v_i^2 / (m_1 + m_2)$$

If $m_2 \neq 0$, then this is less than $KE$. So the collision is, in general, inelastic.

The real usefulness of the notion of kinetic energy is shown in the following (extremely useful) theorem:

**Theorem:** If two particles collide elastically, and the sum of their momenta is 0, then their speeds before the collision are equal to their speeds after it.

This means that if masses $m_1$ and $m_2$ come in with speeds $v_1$ and $v_2$, then they go out with speeds $v_1$ and $v_2$, as long as their collision is elastic, and their total momentum is 0. See the picture below.

This theorem isn't very hard to prove. We'll refer to the diagram above. Let $v_1$ and $v_2$ be the particles' speeds after the collision. Since the total momentum is 0, the particles' momenta must be equal and opposite, before and after the collision. This can only happen if:

$$m_1 v_1 = m_2 v_2$$

$$m_1 u_1 = m_2 u_2$$

These two equations tell us that: $v_1 = m_2 v_2 / m_1$; and $u_1 = m_2 u_2 / m_1$. 
The total kinetic energy before the collision is:

\[ KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \]

The kinetic energy after the collision is:

\[ KE = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 \]

Because the collision is elastic, these two quantities are equal. Therefore:

\[ m_1v_1^2 + m_2v_2^2 = m_1u_1^2 + m_2u_2^2 \]

Now, we substitute in our values for \( v_1 \) and \( u_1 \) to get:

\[ m_1 \left( \frac{m_2v_2}{m_1} \right)^2 + m_2v_2^2 = m_1 \left( \frac{m_2u_2}{m_1} \right)^2 + m_2u_2^2 \]

\[ v_2^2 \left( \frac{m_2}{m_1} + \frac{m_2}{m_2} \right) = u_2^2 \left( \frac{m_2}{m_1} + \frac{m_2}{m_2} \right) \]

\[ v_2 = u_2 \]

The result \( v_1 = u_1 \) can be gotten the same way, or by substituting \( v_2 = u_2 \) back into one of the earlier equations. Q.E.D.

**HOMEWORK**

1. Two masses, \( m_1 \) and \( m_2 \), are stuck together with a piece of plastic explosive and falling under normal Earth gravity, as in the diagram below. When their height is \( h \), and their downward speed is \( v \), the charge between them explodes. Immediately, \( m_2 \)'s downward speed jumps to \( v_0 > v \). How long after the explosion does \( m_1 \), hit the ground? \( m_2 \)?

   \[ \begin{array}{c}
   m_1 \\
   m_2 \\
   \end{array} \]

2. Two particles---one of mass \( m \), the other of mass \( 2m \)---collide at right angles and stick together. The conglomerate moves off, making a \( 45^\circ \) angle with each particle's original direction. What fraction of this system's original kinetic energy is lost in the collision?

   \[ \mathcal{E} \]

3. A particle of mass \( m \) is moving in a circle of radius \( R \), at constant frequency \( \omega \) (\( \omega \) is in radians per second). What is \( m\frac{\omega^2}{2} \) kinetic energy?

4. A particle of mass \( m \) and speed \( v \) collides elastically with a particle of mass \( 2m \). The lighter particle rebounds back in the direction in which it came. What are the speeds of the two masses after the collision?

   \[ \begin{array}{c}
   \downarrow \text{V} \\
   \end{array} \]

   \[ \begin{array}{c}
   m \leftarrow v' = \, ?, \quad \downarrow \text{V}' = \, ? \\
   \end{array} \]

   \[ \begin{array}{c}
   m \\
   \end{array} \]
1. Mass $m_1, m_2, \ldots, m_n$ with velocities $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ collide together to form a big mass $M$. What is $M$'s velocity? (in terms of $m_1, m_2, \ldots$)

2. A mass $m_1$ with speed $\vec{v}_1$ collides elastically with a stationary mass $m_2$. If $m_1 = m_2$, what are the masses' speeds after the collision?

3. Two bodies, each of mass $m$, are moving as in the diagram below. Their respective speeds are (as indicated) $\vec{v}$ and $\vec{u}$. They collide at the point $X$ and stick, and begin to move due north. Find $\vec{u}$, and the masses' speed after the collision.

4. Do problem 3 again, but with the body on the right having mass $2m$.
The Center-of-Mass Frame

An evil physics instructor has given us the following problem. Two masses, \( m \) and \( 2m \), approach each other with equal speeds \( v \). They collide elastically, and \( m \) is deflected by an angle of \( 30^\circ \). The evil instructor has diagrammed the situation for us below:

Before

\[ \vec{v} \rightarrow \vec{v}_1 \]

After

\[ \vec{u}_1 \rightarrow \vec{u}_2 \]

\[ \theta' \]

We must find \( u_1 \) and \( u_2 \) (the particles' speeds after the collision), and \( \theta' \) (the angle through which \( 2m \) is deflected).

By using conservation of momentum, and the fact that the kinetic energy is conserved, we could do this. It would be a messy job. Instead of doing it the direct, messy way, we will devise a trick: the CM (center-of-mass) frame.

The idea behind the CM-frame is this: If the total momentum of the above system were zero, then the problem would be easy; \( \theta' \) would equal \( 30^\circ \), and \( u_1 \) and \( u_2 \) would be \( v \). But the momentum clearly isn't zero; at least, not in our present reference frame. We will find a reference frame in which it is zero—the CM-frame—and see what the collision looks like from there. It will be easy, from what was proved in Handout #13. When we put the views of the collision as seen in the stationary ("lab") frame and the CM-frame together, the answers to our problem will come easily.

I said that the collision in the CM-frame will be easy to do. That will be true only if the collision (in that frame) is elastic. Now, it is reasonable that it should be. If it were not elastic, then some of the masses' kinetic energy would have been turned into other forms of energy, such as heat and noise. But if the masses heat up or make noise in one frame, then they do it in all of them. Therefore, if they lose energy in one frame, they lose it in all frames. Since there is at least one frame in which no energy is lost—the lab frame—we can say, for physical reasons, that no energy is lost in any frame; the collision is elastic in all reference frames.

That is a sloppy sort of proof. Fortunately, we can prove the "conservation of elasticity" mathematically. We will prove the following general theorem:

**Theorem:** If two masses collide elastically in one frame of reference, then they collide elastically in all frames.

We will look at a perfectly general collision, such as the one below. The masses and velocities are as marked. We will denote their speeds by \( v_i \), \( u_i \), etc., rather than \( ||\vec{v}_i|| \).

Before

\[ \vec{v}_1 \rightarrow \vec{v}_2 \]

After

\[ \vec{u}_1 \rightarrow \vec{u}_2 \]

Because this collision is elastic, we have the equation:

\[
\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2
\]
Because momentum is conserved, we also know that:

\[ m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{u}_1 + m_2 \vec{u}_2 \]

The diagram on page 40 is how things look from the lab frame. How do they look from another frame, one with a velocity \( \vec{V} \)?

Rather than having velocities \( \vec{v}_1, \vec{v}_2, \vec{u}_1, \) and \( \vec{u}_2 \), the masses will have velocities \( \vec{v}_1', \vec{v}_2', \vec{u}_1', \) and \( \vec{u}_2' \), which will equal:

\[ \vec{v}_1' = \vec{v}_1 - \vec{V} \]
\[ \vec{v}_2' = \vec{v}_2 - \vec{V} \]
\[ \vec{u}_1' = \vec{u}_1 - \vec{V} \]
\[ \vec{u}_2' = \vec{u}_2 - \vec{V} \]

And the collision will look like this:

**Before**

\[ \begin{array}{c}
\downarrow \vec{v}_1 \\
m_2 \\
\uparrow \vec{v}_2
\end{array} \]

**After**

\[ \begin{array}{c}
\downarrow \vec{v}_1' \\
m_1 \\
\uparrow \vec{v}_2' \\
m_2
\end{array} \]

How did I get those equations? Consider this problem: If my velocity is \( \vec{V} \) and yours is \( \vec{U} \), what velocity do you seem to have, from my point of view? If you were standing still, you would seem to have a velocity \( \vec{U} - \vec{V} \). But your velocity is \( \vec{U} \). Your "relative velocity" will be equal to:

\[ \vec{v} = \text{("real" velocity) + (velocity "caused" by my motion)} \]
\[ \vec{v} = \vec{U} - \vec{V} \]

The truth of this is easier to see in an example. If I pass a telephone pole, going 50 mph due east, the telephone pole (if I am not very smart) will look to be going 50 mph due west. Now, if the pole were to shoot up like a rocket, then the velocity it would appear to have would be its upward velocity added (as a vector) to that 50 mph due west: \( \vec{U} - \vec{V} \). If you work out some examples of your own, then you will understand the formula better.

Energy

In the moving frame, the kinetic energy before the collision, \( KE_B \), is:

\[ KE_B = \frac{1}{2} m_1 (v_1')^2 + \frac{1}{2} m_2 (v_2')^2 \]

The energy after the collision, \( KE_A \), is:

\[ KE_A = \frac{1}{2} m_1 (u_1')^2 + \frac{1}{2} m_2 (u_2')^2 \]
The vector handout (#6) says that \((v_i \cdot \hat{v} = v_i \cdot \hat{v}_i\). According to the equations on page 41, this equals:

\[
(v_i \cdot \hat{v})^2 = (v_i \cdot \hat{v}) \cdot (v_i \cdot \hat{v}) = v_i \cdot \hat{v} - 2(v_i \cdot \hat{v}) + \hat{v}_i \cdot \hat{v}_i = v_i^2 - 2(v_i \cdot \hat{v}) + \hat{v}_i^2
\]

We get similar expressions for the other speeds. Our expression for \(KE_B\) becomes:

\[
KE_B = \frac{1}{2} m_i v_i^2 - m_i (v_i \cdot \hat{v}) + \frac{1}{2} m_i \hat{v}_i^2
\]

\[
+ \frac{1}{2} m_2 v_2^2 - m_2 (v_2 \cdot \hat{v}) + \frac{1}{2} m_2 \hat{v}_2^2
\]

The new equation for \(KE_A\) is just as monstrous:

\[
KE_A = \frac{1}{2} m_i u_i^2 - m_i (u_i \cdot \hat{v}) + \frac{1}{2} m_i \hat{v}_i^2
\]

\[
+ \frac{1}{2} m_2 u_2^2 - m_2 (u_2 \cdot \hat{v}) + \frac{1}{2} m_2 \hat{v}_2^2
\]

The collision will be elastic if \(KE_B - KE_A = 0\). This difference equals:

\[
KE_B - KE_A = \frac{1}{2} m_i v_i^2 - m_i (v_i \cdot \hat{v}) + \frac{1}{2} m_i \hat{v}_i^2 + \frac{1}{2} m_2 v_2^2 - m_2 (v_2 \cdot \hat{v}) + \frac{1}{2} m_2 \hat{v}_2^2 - \]

\[
\left[ \frac{1}{2} m_i u_i^2 - m_i (u_i \cdot \hat{v}) + \frac{1}{2} m_i \hat{v}_i^2 + \frac{1}{2} m_2 u_2^2 - m_2 (u_2 \cdot \hat{v}) + \frac{1}{2} m_2 \hat{v}_2^2 \right]
\]

The terms which have the \(X\)'s between them cancel, and we are left (after some regrouping) with:

\[
\left[ \frac{1}{2} m_i v_i^2 - m_i (v_i \cdot \hat{v}) + \frac{1}{2} m_i \hat{v}_i^2 \right] - \left[ \frac{1}{2} m_i u_i^2 - m_i (u_i \cdot \hat{v}) + \frac{1}{2} m_i \hat{v}_i^2 \right]
\]

Look at the equation on the bottom of page 40. The quantity in the first pair of brackets is the difference of the before and after kinetic energies, as seen in the lab frame. This is zero. Our expression becomes (after still more regrouping, using the distributivity and scalar-multiplication properties of the dot product):

\[
\left[ m_i (\hat{v} \cdot \hat{v}) + m_2 (\hat{v} \cdot \hat{v}) \right] \cdot \hat{v} = 0
\]

Now look at the equation at the top of page 41. The vector in the brackets is the difference of the before and after momenta—which is the zero vector. The expression simplifies further to:

\[
0 \cdot \hat{v} = 0 \cdot \hat{v}
\]

The theorem is proved.

This theorem tells us that it's all right to go looking for the magical CM-frame. We still have to find it. As usual, we will find a rule for the general case, and then apply that rule to our particular problem.
Suppose that we have a system of masses $m_1, m_2, \ldots, m_n$, with velocities $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ (as seen in the lab frame). The total momentum of the system (as seen in this frame) is:

$$\vec{p} = \sum_{i=1}^{n} m_i \vec{v}_i$$

If we are in a frame that has velocity $\vec{v}$, the the mass $m_i$ will (in our frame) have a velocity: $\vec{v}_i - \vec{v}$. For us, the total momentum will be:

$$\vec{p}' = \sum_{i=1}^{n} m_i (\vec{v}_i - \vec{v})$$

Now we set $\vec{p}'$ equal to 0 and solve for $\vec{v}$: velocity of CM-frame.

$$\vec{v} = \frac{\sum_{i=1}^{n} m_i (\vec{v}_i - \vec{v})}{\sum_{i=1}^{n} m_i}$$

In English, $\vec{v} = \frac{\text{(sum of the momenta)}/(\text{sum of the masses})}$.

Having done all this work, we can now do the E.P.I.'s (Evil Physics Instructor) problem easily. Not only this, but if any other E.P.I. should give us (you) a similar collision problem, then we (you) have a way by which we can do it in $2x10^6$ shakes of a comet's tail.

The two masses start out on the x-axis. Therefore, we don't have to worry about their y-momenta when we calculate $\vec{v}$. Also, we can drop the arrows, because the total momentum only has an x-component. If we let motion toward the right (see the picture on page 104) be positive, then:

$$v = \frac{(mv - 2mv)}{(m + 2m)}$$

$$v = -\frac{v}{3}$$

Let $v_1$ and $v_2$ be the velocities of $m_1$ and $2m_2$ as seen in the CM-frame. By the equations on page 104:

$$v_1 = v - (-v/3)$$

$$= 4v/3$$

$$v_2 = -v - (-v/3)$$

$$= -2v/3$$

These are their x-velocities before the collision, as seen in the CM-frame. Their speeds are just the absolute values of these. Since the collision is elastic and their total momentum is zero, they will rebound in opposite directions, with the same speeds at which they came in. In the CM-frame, therefore, the collision will look like:

$+ you$
where the angle $\alpha$ is as yet unknown. To find $\alpha$, we translate the "after" picture back into the lab frame. This is done by adding $\vec{V}$ (as a vector) to velocities (vectors) that the masses have in the CM-frame.

\textit{After (CM)}

The picture on the right has to be identical to the "after" picture at the top of page 40. That picture is shown below. Since the corresponding parts have to be equal (for they are two pictures of the same thing seen in the same frame), the angle $\theta$ must be $30^\circ$. Therefore, $\tan \theta = \tan 30^\circ = \sqrt{3}/3$. But we can calculate $\tan \theta$ from the above diagram. It is:

$$
\tan \theta = \frac{(\sqrt{3}/3) \sin \alpha}{(\sqrt{3}/3) \cos \alpha - \sqrt{3}/3}
$$

$$
= \sin \alpha / (\cos \alpha - 1)
$$

The square of this must be $1/3$. If we square it, we get:

$$
\frac{1}{3} = \frac{16 \sin^2 \alpha}{(16 \cos^2 \alpha - 8 \cos \alpha + 1)}
$$

$$
= \frac{16(1 - \cos^2 \alpha)}{(16 \cos^2 \alpha - 8 \cos \alpha + 1)}
$$

$$
16 \cos^2 \alpha - 8 \cos \alpha + 1 = 16 - 16 \cos^2 \alpha
$$

$$
64 \cos^2 \alpha - 8 \cos \alpha - 17 = 0
$$

We use the quadratic formula to get that $\cos \alpha = 0.922$ or $-0.797$. The corresponding values for $\alpha$ and $\sin \alpha$ are: $22.7^\circ$, $0.387$; $112.3^\circ$, $-0.387$. Which set do we pick? The quantity:

$$
\sin \alpha / (\cos \alpha - 1)
$$

has to be positive, and equal to $\sqrt{3}/3$. The second set of values doesn't make it. Therefore:

$$
\alpha = 22.7^\circ
$$

$$
\cos \alpha = 0.922
$$

$$
\sin \alpha = 0.387
$$
This immediately gives us $u_i$. By the Pythagorean Theorem:

$$u_i^2 = \left(\frac{\sqrt{3}v}{3}\right)^2 + \left(\frac{\sqrt{3}v}{3}\right)^2$$

$$= (\frac{v^2}{9})^2 \left[\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right]$$

$$= (\frac{v^2}{9})^2 (9,62)$$

$$u_i = 1.03v$$

And this gives us $u_2$ just as quickly. Because the collision is elastic:

$$\frac{1}{2}mv^2 + \frac{1}{2}2mv^2 = \frac{1}{2}mu_i^2 + \frac{1}{2}mu_2^2$$

$$(3/2)v^2 - \frac{1}{2}u_i^2 = u_2^2$$

$$u_2^2 = v^2(3/2 - 9.62/18)$$

$$u_2 = 0.983v$$

The total $y$-momentum before the collision is zero. The total $y$-momentum after the collision is zero. According to the picture on page 14, this will not be true unless:

$$2mu_2\sin\theta = mu_1\sin30^\circ$$

$$\sin\theta = (\sin30^\circ)(u_1/u_2)(\frac{1}{3})$$

$$= (\frac{1}{3})(1.03/0.983)$$

$$\theta = 15.19^\circ$$

We have solved the problem, and the E.P.I. is foiled again. "Curses!" he cries. "Just wait until angular momentum!"

**Homework**

1. In the system below, $m_1 = m_2$; $m_2$ is initially at rest, and is struck by $m_1$, as shown. The collision is elastic. If $m_1$ has initial speed $v$, and $\theta = 30^\circ$, what are $u_1$, $u_2$, and $\alpha$?

   ![Diagram](before.png)

   ![Diagram](after.png)

2. A particle of mass $m$ and speed $v$ collides elastically with a particle of mass $m_2 = 3m$. The mass $m_2$ goes off at an angle of $45^\circ$, as in the diagram below. What are $\theta_1$, $u_1$, and $u_2$? (Answer: $\theta_1 = 71.57^\circ = \tan^{-1}3$; $u_1 = v/\sqrt{10}$; $u_2 = (v/4)(1/2)$.)

   ![Diagram](before.png)

   ![Diagram](after.png)
3. Mass $m$ and $2m$ are approaching each other with speeds $5v$ and $v$, as in the diagram below. They collide elastically, and $m$ is deflected by an angle of $15^\circ$. Find $\theta_1$, and the speeds $u_1'$ and $u_2'$. (Answer: $\theta_1 = 85.06^\circ$; $u_1' = 1.64v$; $u_2' = 1.65v$.)

\[ \begin{array}{c}
\text{Before} \\
5v \rightarrow m & 2m \leftarrow v \\
\end{array} \begin{array}{c}
\text{After} \\
\theta_1 \quad u_1' \quad u_2' \quad 45^\circ \\
\end{array} \]

4. The collision represented below is impossible. Why?

\[ \begin{array}{c}
\text{Before} \\
m \rightarrow m \\
\end{array} \begin{array}{c}
\text{After} \\
m \quad 60^\circ \quad 60^\circ \\
\end{array} \]
Work and Potential Energy

In classical physics, there are two types of energy: potential and kinetic. Kinetic energy is the energy of bodies in motion. Potential energy is the ability to put bodies into motion. Classically, all forms of energy—heat, electrical energy, etc.—are forms of one or the other of these two. Heat, for instance, is the result of molecular kinetic energies.

These two forms of energy can be used to do work. When a constant force \( \vec{F} \) is applied over a vector \( \vec{L} \), then the work done by the force is defined as:

\[
W = \vec{F} \cdot \vec{L}
\]

Work = force applied x distance. This is shown in the diagram below.

Although the force is rather large, only a small part of it—\( F \cos \theta \)—is doing any work.

Problem: If you move a weight horizontally, how much work do you do? None: the force applied, to support the weight, is vertical. The vector over which it is applied is horizontal. Their dot product is zero.

A force can also act over a curved path, and it need not be constant. In that case, we break up the path into many little vectors, denoted \( \overset{\leftrightarrow}{dS} \), as in the picture to the left. We evaluate the quantity \( \vec{F} \cdot \overset{\leftrightarrow}{dS} \) at every little \( \overset{\leftrightarrow}{dS} \) (where, remember, \( \overset{\leftrightarrow}{dS} \) might be varying). Then we add all the little \( \vec{F} \cdot \overset{\leftrightarrow}{dS} \)'s together. This quantity:

\[
\sum \vec{F} \cdot \overset{\leftrightarrow}{dS}
\]

is, if the \( \overset{\leftrightarrow}{dS} \)'s are small, a good approximation to the work done by the force. The exact value is found by taking the limit as the \( \overset{\leftrightarrow}{dS} \)'s become infinitesimal. Then we say that the work done by the force is:

\[
W = \int \vec{F} \cdot \overset{\leftrightarrow}{dS}
\]

where the integral is taken over the path.

Problem: How much work is done by displacing a spring of constant \( k \) by an amount \( x \)? At any point \( x_0 \), the force that must be applied to move the spring is \( kx \). The little element of our path—which corresponds to the \( \overset{\leftrightarrow}{dS} \)—is a \( dx \). Since the force is parallel to the displacement, and in the same direction, \( \vec{F} \cdot \overset{\leftrightarrow}{dS} \) just equals \( kx \, dx \). Then the work integral is:

\[
W = \int kx \, dx = \frac{1}{2} kx^2
\]

This work is the potential energy that is stored in the spring.

Problem: How much work is done by lifting a mass \( m \) a height \( h \) off the ground? Here the force that must be applied is \( mg \), upward. If \( dx \) denotes
our path element, then $F \cdot ds$ here will equal: $mg \cdot$ little change in height $= mg \cdot dh$. The work integral is:

$$W = \int_{h_0}^{h} mg \cdot dh = mgh_0$$

This is represented below.

We cannot, as with the spring, define the potential energy of a mass $m$ at a height $h_0$ to be equal to $mgh_0$. Because, what is $h_0$? It is the mass's height above the Earth's surface. But the Earth's surface is not smooth--so what is its "real" level?

We don't worry about this. When we do problems, we just define some height--usually the local ground level--to be our zero height, and define potential energies and work relative to that height.

Energy is a useful concept, because, like momentum, it is often conserved. At least, it is conserved to a good enough approximation for us to do problems using it. The Law of the Conservation of Energy says that, unless energy is leaving a system--in the form of heat, noise, etc.--or entering it, then the system's total energy (potential + kinetic energy) remains a constant.

A simple example of this is a falling body. If its mass is $m$, then when its height is $h$, its potential energy is $mgh$. If it is falling with a speed $v$, then its kinetic energy is $\frac{1}{2}mv^2$. Therefore, its total energy is:

$$E = mgh + \frac{1}{2}mv^2$$

Let's differentiate this with respect to time. We get:

$$\frac{dE}{dt} = mg\left(\frac{dh}{dt}\right) + mv\left(\frac{dv}{dt}\right)$$

At any time, $\frac{dh}{dt} = -v$; whereas $\frac{dv}{dt} = g$ (since $\frac{dv}{dt}$ is the change of the speed with time). Then the expression becomes:

$$\frac{dE}{dt} = -mgv + mvg$$

$$= 0$$

Hence, $E$ is constant with time.

Problem: A projectile of mass $m$ is fired into the air. It reaches a height $h$. What was its muzzle speed, $v$? When the shell is fired, it has no potential energy. Its kinetic energy is $\frac{1}{2}mv^2$. So:

$$E = \frac{1}{2}mv^2$$

When it gets to $h$, it will have no kinetic energy, but a potential energy equal to $mgh$. By the conservation law, this must equal the energy it had when it was fired:

$$mgh = \frac{1}{2}mv^2$$

$$v = \sqrt{2gh}$$

This is the same answer we get with kinematics (check it).
Problem: In the system below, everything slides without friction. If \( m_1 \) slides down \( m_2 \), starting from the top, as shown, what are the masses' final speeds? Before \( m_1 \) slides down, the system's total energy is equal to \( m_1 \)'s potential energy. This is just:

\[
E = m_1 gh
\]

In the final state, all the energy is kinetic, and equals:

\[
E = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2
\]

But by conservation of momentum:

\[
m_1 u_1 = m_2 u_2
\]

\[
u_1 = \left( \frac{m_2}{m_1} \right) u_2
\]

So we can re-write the final kinetic energy as:

\[
E = \frac{1}{2} m_1 \left( \frac{m_2}{m_1} u_2 \right)^2 + \frac{1}{2} m_2 u_2^2
\]

\[
= \frac{1}{2} u_2^2 \left[ \frac{m_2^2}{m_1} + m_2 \right]
\]

This has to equal the initial energy.

\[
m_1 gh = \frac{1}{2} u_2^2 \left[ \frac{m_2^2}{m_1} + m_2 \right]
\]

\[
u_2 = \left[ \frac{2gh}{\frac{m_2^2}{m_1} + m_2} \right]^{\frac{1}{2}}
\]

\[
u_1 = \left( \frac{m_2}{m_1} \right) \left[ \frac{2gh}{\frac{m_2^2}{m_1} + m_2} \right]^{\frac{1}{2}}
\]

This shows the usefulness of the concept of energy. This problem was done with relative ease. Imagine trying to do it with forces.

**Homework**

1. In the system below, the spring has constant \( k \) and is massless. The masses are initially at rest and the spring is compressed an amount \( x \). The surface is frictionless. If the masses are simultaneously released, what will be their speeds?

   **Before**

   \[
   \begin{array}{c}
   m_1 \ \ \ \ \ m_2
   \end{array}
   \]

   **After**

   \[
   \begin{array}{c}
   m_1 \ \ \ \ \ m_2
   \end{array}
   \]

2. What is the work/revolution required to keep a mass \( m \) going in a circle of radius \( R \), at frequency \( \omega \)?
3. Before Darth Vader turned to evil, he liked to play on swings. Below is a drawing of him, done by his mother, Ma Vader. The swing rope is of length $L$, and he swings through an angle $2\theta$, as shown. What is Darth's speed at the bottom of his swing? Assume that normal gravity is acting.

4. The system below is in equilibrium. Write the mass's total energy as a function of $x$ and show that $dE/dx = 0$; i.e., the energy is at a minimum.
In many physical situations, a particle's potential energy can be a function of its position in space. Examples of this are gravitational potential energy and the potential energy of a particle that is attached to the end of a spring. But we do not need to be limited to these few concrete situations. Instead, we can look at the nature of potential energy functions in general, and how they can give us information about the motion of particles.

For now, let's work in one dimension. A particle of mass $m$, let us say, is free to move along the $x$-axis. There is no friction; nothing that might give the particle energy or take it away. The only thing that is affecting the particle's motion is the fact that its potential energy is a function of its position. If it is at the point $x$, then its potential energy is $U(x)$.

A good question is: what is the force that is acting on the particle at $x$?

Since it is moving freely, with nothing to lessen or increase its total energy, the quantity:

$$E = U(x) + \frac{1}{2}mv^2$$

($=$ potential + kinetic energy) must remain constant with time. Therefore:

$$\frac{dE}{dt} = 0 = \frac{d(U(x))}{dt} + mv\frac{dv}{dt}$$

According to the chain rule, the expression marked $A$ equals:

$$\frac{dx}{dt} = \frac{dU}{dx}$$

Therefore:

$$0 = (dx/dt)(dU/dx) + mv(dv/dt)$$

Now, $dx/dt$ is $v$, the mass's velocity; and $dv/dt$ is $a$, its acceleration. Therefore:

$$0 = v(dU/dx) + ma$$

The last equation has to be true for all possible values of $v$; e.g., when $v \neq 0$. This cannot be true unless:

$$0 = dU/dx + ma$$

$$ma = -dU/dx$$

$$F = -dU/dx$$

where $F$ is the force on the particle at the point $x = F(x)$.

Example: A mass on a spring of constant $k$. $U(x) = \frac{1}{2}kx^2$. Then:

$$F(x) = -dU/dx = -kx$$
Example: A falling body of mass m. Here $U(h) = \text{potential energy as a function of height } = mgh$. And:

$$F(h) = -\frac{dU}{dh} = -mg$$

These (I hope) illustrate the idea.

We think of this $U(x)$ and its associated $F(x)$ as being stable structures, like masses and springs. We call them fields. The fields acting in a region of space determine the forces that act on particles in that region. Examples of fields are electric, magnetic, and gravitational fields. A spring is a sort of "field" because it determines the forces that act on any mass tied to its end.

Fields only determine the forces that are acting on free particles: particles which are being acted on by nothing but the field. When I raise my hand against the force of gravity, my hand cannot be considered a free particle, because my arm is acting on it. Gravity isn't the only force acting on it.

The relation $\Delta x$ on page 51 can in some sense be reversed. Not only can we get $F(x)$ from $U(x)$, we can almost get $U(x)$ from $F(x)$. The Fundamental Theorem of Calculus says that:

$$(*) \quad U(x) - U(a) = -\int_{a}^{x} F(x) \, dx$$

Therefore, if we know the value of $U(x)$ anywhere, we know it everywhere, wherever the field exists.

Problem: -- force field $F(x)$ is acting on the x-axis. If a particle has mass $m$ and is not at $x = 0$, then the force acting on it equals $-m/x^2$. A mass $m$ starts from the point $x = 1$ with a velocity of $v > 0$. At the point $x = L$ it stops and begins to fall back toward the origin. What is $v$?

We'll let $U(x)$ be $m$'s potential energy as a function of $x$. When $m$ is at $x = 1$, its total energy is:

$$E = U(1) + \frac{1}{2}mv^2$$

When it stops at $x = L$, its kinetic energy is zero. Its total energy then is:

$$E = U(L)$$

These have to be equal. Therefore:

$$0 = E' - E = U(L) - U(1) - \frac{1}{2}mv^2$$

$$U(L) - U(1) = \frac{1}{2}mv^2$$

Equation $(*)$ says that:

$$U(L) - U(1) = -\int_{1}^{L} \frac{m}{x^2} \, dx = \int_{1}^{L} \frac{m}{x^2} \, dx = m - m/L$$

"which means that:

$$\frac{1}{2}mv^2 = m - m/L$$

$$v = \left(2 - \frac{2}{L}\right)^{1/2}$$
The last equation can be turned around, to give us \( L \) in terms of \( v \):

\[
L = \frac{2}{(2-v^2)}
\]

As \( v \) increases to \( \sqrt{2} \), \( L \) increases without limit. If the mass's initial velocity is greater than or equal to \( \sqrt{2} \), it will never fall back to the origin. The square root of two is the escape velocity for this force field, from the point \( x=1 \).

**HOMEWORK:** If in the above problem, the force on a mass \( m \) had been:

\[
F(x) = -\frac{2mx}{(x^2+1)^2}
\]

what would the escape velocity have been, for a particle starting from \( x=0 \)?

When we are working in three dimensions, the problem of finding the force from the potential function is a little more complicated. In three dimensions, our potential function is not \( U(x) \), but:

\[
(I) \quad U = U(x,y,z)
\]

where \((x,y,z)\) are the co-ordinates of a point. Here the force is a vector with three components:

\[
(II) \quad F(x,y,z) = F_x(x,y,z)\hat{i} + F_y(x,y,z)\hat{j} + F_z(x,y,z)\hat{k}
\]

The \((x,y,z)\)'s mean that the force is a function of where one is in space. An example of such a force is the gravitational force exerted by a mass. The force changes direction and magnitude as one moves from place to place around the mass.

The three-dimensional problem is solved by first looking at it as three one-dimensional problems. We pretend that our mass \( m \) (whose potential function is given by \((I)\)) has had its motion restricted to a straight line: a line along which \( y \) and \( z \) are constant. If the mass stays on that line, then its potential will just be a function of \( x \). The force pushing it along that line will be equal to:

\[
F(x) = -\frac{dU}{dx}
\]

This \( F(x) \) has to equal \( F(x,y,z) \)—as long as we are on that line. We can do this same thing with every line \( L \) for which \( y \) and \( z \) are constant. Let us fix \( y \) and \( z \) at \( y_0 \) and \( z_0 \). Every point on that line will have co-ordinates of the form \((x,y_0,z_0)\). And the force along that line can be gotten by the above equation. Hence:

\[
F(x,y_0,z_0) = -\frac{dU(x,y_0,z_0)}{dx}
\]

where, when we take the derivative on the right, we treat \( y_0 \) and \( z_0 \) as constants. A derivative like this, where all the variables but one are treated as constants, is called a partial derivative. And it is written a strange way. The conventional (i.e., correct) way to write the above equation is:

\[
F(x,y,z) = -\frac{dU(x,y,z)}{dx}
\]
The expressions for $F_y$ and $F_z$ are similar:

$$F_y(x,y,z) = -\partial U(x,y,z)/\partial y$$
$$F_z(x,y,z) = -\partial U(x,y,z)/\partial z$$

Often, when we write partial derivatives, we don't write the "$(x,y,z)$"; the last three equations can be written:

$$F_x(x,y,z) = -\partial U/\partial x$$
$$F_y(x,y,z) = -\partial U/\partial y$$
$$F_z(x,y,z) = -\partial U/\partial z$$

When we substitute these back into (II), we get the force:

$$\vec{F}(x,y,z) = -(\partial U/\partial x)\hat{i} - (\partial U/\partial y)\hat{j} - (\partial U/\partial z)\hat{k}$$

Problem: If $U(x,y,z) = x^2 + y^2 + z^2$, what is $\vec{F}(x,y,z)$? We find the partial derivatives, $\partial U/\partial x$, $\partial U/\partial y$, $\partial U/\partial z$. To find $\partial U/\partial x$, differentiate:

$$x^2 + y^2 + z^2$$

with respect to $x$, treating $y$ and $z$ as constants. We get:

$$\partial U/\partial x = 2x$$

The others are similar: $\partial U/\partial y = 2y$; $\partial U/\partial z = 2z$. The force equals:

$$\vec{F}(x,y,z) = -2(x\hat{i} + y\hat{j} + z\hat{k})$$
$$\quad = -2\vec{r}$$

where $\vec{r}$ is the position vector of the point $(x,y,z)$.

Problem: What if $U(x,y,z) = x^2y^2z^2$? This isn't hard, because:

$$\partial U/\partial x = 2xy^2z^2$$
$$\partial U/\partial y = 2x^2yz^2$$
$$\partial U/\partial z = 2x^2y^2z$$

The force is:

$$-2xy^2z^2\hat{i} - 2x^2yz^2\hat{j} - 2x^2y^2z\hat{k}$$

You may ask, Is it possible, given $\vec{F}$, to find $U$? Sometimes. But that problem is much more complicated. What is worse, given an $\vec{F}$, there need not exist a corresponding $U$.

**HOMEWORK**

1. Find the $\vec{F}(x,y,z)$ for each $U$ below.

   a. $1/(x^2 + y^2 + z^2)$  
   c. $x + y + z$
   
   b. $(\sin x)(\sin y)(\sin z)$  
   d. $3xy^2 - 16x^2y + 16z^2$
2. Han Solo is flying his ship dangerously close to the Death Star. Suddenly, a tractor beam comes on, which creates a potential field around the Star: \( U(x, y, z) = 2x^2 - 3y^2 + z^2 + 9x + 1y - 7z \). (The center of the spherical Death Star has coordinates \((0, 0, 0)\).) What are the coordinates of the point to which Han should go, if he wants to escape the Force of the beam?

3. For a particle on the \( x \)-axis, \( U(x) = x^2 - x^2 \). If the particle’s total energy gets above an amount \( E \), the particle will go on forever toward \( x = +\infty \). What is \( E \)?

4. For another particle, \( U(x) = -x^2 \). What is the force on the particle when it is at \( x = 0 \)? If the particle is at rest there, and I nudge it, what will happen? Will it go back to \( x = 0 \)?

5. In the \( xy \)-plane, \( U(x, y) = (x-y)^2 \). There is a set of points on which the force is \( 0 \). What is that set?
\[ U(xy) = (x^2 + y^2)^{\frac{1}{2}} \]

\[ F(r) = 2\hat{r} \]

\[ \vec{v}(0) = \hat{r} \]

\[ Z = (x^2 + y^2)^{\frac{1}{2}} \]
\[ u(x, y) = x^2 + y^2 \]

\[ \frac{\partial u}{\partial t}(0) = 2 \]

\[ \frac{\partial u}{\partial x}(0) = \frac{1}{2} \]

\[ z = x^2 + y^2 \]
\[ U(x, y) = \frac{-1}{(x^2 + y^2)^{\frac{3}{2}}} \]
\[ \mathbf{F}(0) = 2 \hat{i} \]
\[ \mathbf{V}(0) = \frac{1}{2} \hat{j} \]

\[ Z = \frac{-1}{(x^2 + y^2)^{\frac{3}{2}}} \]
$U(x,y) = \frac{-1}{x^2 + y^2 + 1}$

$\vec{r}(0) = \vec{i}$

$\vec{V}(0) = \frac{1}{2} \vec{j}$

$Z = \frac{-1}{x^2 + y^2 + 1}$
\( u(x,y) = \sin(\sqrt{x^2 + y^2}) \)

\[ p(0) = \frac{\pi}{2} \]
\[ \dot{x}(0) = \frac{1}{2} \dot{y} \]

\[ z = \sin(\sqrt{x^2 + y^2}) \]
\[ U(x, y) = \frac{1}{(x^2 + y^2)^{\frac{1}{2}}} \]
\[ \vec{F}(0) = \hat{y} \]
\[ \vec{V}(0) = \hat{z} \]

\[ Z = (x^2 + y^2)^{-\frac{1}{2}} \]
1. On the planet Kabro (Darth Vader's home world), the potential energy of a mass \( m \) at a height \( h \) is not \( mgh \), but \( kmh^2 \), where \( k \) is a constant. If a projectile is fired from ground level, what initial speed must it have to reach a height \( h \)?

2. The Death Star is a black, evil-looking sphere of radius \( R \). It is protected by a force field: For a mass \( m \) that is outside the Star, its potential energy is \( A^2 m/(R-r)^2 \); where \( A \) is a constant, and \( r \) is the distance between the mass and the Star's center. A daring penguin, from a very great (i.e., infinite) distance away, hurls his suitcase—which contains an old ARM warhead—directly at the Star. How big must the case's initial speed be, in order for it to get to within \( R/2 \) of the Star's surface?

3. In the xy-plane, \( U(x,y) = xy(x-a)(y-b) \). (\( a \) and \( b \) are constants). Prove or disprove that the point \((a/2, b/2)\) is a point of equilibrium.

4. Find the \( \vec{F}(x,y) \) for each \( U(x,y) \) below:
   a. \( x^2 - y^2 \)
   b. \( \sin(x^2 + y^2) \)
   c. \( (x^2 + 1)/(y^2 + 1) \)

5. Extra Credit: Sketch the force vectors in the plane for \( U(x,y) = x^2 - y^2 \).
Gravitation

Two point masses, separated by a distance $R$, exert a force on each other. This force is the gravitational force or gravitation. The force is parallel to the straight line that joins the two masses, and tends to pull them toward one another. Its magnitude is:

\[
\frac{Gm_1m_2}{R^2}
\]

where $m_1$ and $m_2$ are the masses, and $G$ is a constant. This constant can only be found by experiment. It has been found to equal:

\[
G = 6.670 \times 10^{-8} \text{ dyne-cm}^2/\text{gm}^2 = 6.670 \times 10^{-11} \text{ newton-meter}^2/\text{kg}^2
\]

This means that if two weights of one gram each are separated by one centimeter, each will feel a tug toward the other of $6.670 \times 10^{-8}$ dynes.

The manner in which gravitation acts is shown below.

\[
\begin{array}{c}
\vec{F}_1 \\
\Rightarrow \\
R \\
\Rightarrow \\
\vec{F}_2
\end{array}
\]

In order to find out how large bodies attract each other, one breaks up the bodies into many point masses. Then one calculates the force (vector) that each point mass in one body is exerting on each point mass in the other body. The total force acting on a body will be the vector sum of the forces acting on all its point masses. As the number of point masses becomes infinite, this vector sum becomes an integral.

We will not have to work such an integral. When two bodies are separated by a distance that is very large compared to their dimensions (e.g., two Volkswagens a million miles apart), they attract each other almost as though they were point masses. In such cases, equation (1) is good enough for anybody.

For bodies that are fairly close to each other (such as the Earth and the Moon), we will use another fortunate fact (we will not prove it, because it requires the working of a terrible integral). A spherical shell that is of uniform density attracts masses that are outside it as if all of its mass were concentrated at its center. The masses must be outside the shell; the shell exerts no force on a mass that is inside it. And the shell must be of uniform density: no lumens, in other words.

A corollary of this fortunate fact is that a solid sphere, whose density only varies with distance from its center, also attracts objects as if all its mass were concentrated at its center. This is because such a sphere can be thought of as being made of many thin, uniformly dense shells (like the skins on an onion). As with the hollow spheres, this rule only works for objects that are outside the sphere. And the shells of which the sphere is made must be uniformly dense. See the picture to the left.
Problem: "Weigh the Earth. To a very good approximation, the Earth is like one of the spheres just mentioned. Hence we can solve the problem by pretending that all of its mass (denoted \( M \)) is concentrated at its center. The force on a body of mass \( m \) at the Earth's surface must be identical to that which would be exerted by a mass \( M \) which is a distance \( R \) away, where \( R \) is the Earth's radius. Now, this force's magnitude is:

\[
P = \frac{GmM}{R^2}
\]

However, the mass \( m \) is feeling a force of magnitude \( mg \), due to gravity. These forces must be equal. Hence:

\[
mg = \frac{GmM}{R^2}
\]

\[
M = \frac{R^2}{G} = \frac{(6.4 \times 10^8 \text{ cm})^2}{980 \text{ cm/sec}^2}/G = 6.02 \times 10^{17} \text{ gm}
\]

Problem: A satellite is orbiting in a circle about the Earth, of radius \( R \). Its speed is a constant, \( v \). What is \( v \)?

The magnitude of the force needed to hold the satellite in its orbit is:

\[
\frac{mv^2}{R}
\]

where \( m \) is the satellite's mass. This must be supplied by the gravitational force, whose magnitude is:

\[
\frac{GmM}{R^2}
\]

Setting them equal, we get:

\[
\frac{mv^2}{R} = \frac{GmM}{R^2}
\]

\[
v^2 = GM/R
\]

\[
v = \sqrt{\frac{GM}{R}}
\]

For a satellite 200 km above the Earth's surface, \( R = 6600 \text{ km} = 6.6 \times 10^8 \text{ cm} \). Plugging that and the value for the Earth's mass into the last equation gives an orbital speed of \( 780,000 \text{ cm/sec} = 17,137 \text{ mph} \).

Problem: How long is a lunar month? The distance between the Earth and the Moon is about 240,000 miles = \( 3.86 \times 10^{12} \text{ cm} \). With that, the Moon's orbital speed is \( 102,000 \text{ cm/sec} \). In one lunar month it must travel a distance of \( 2\pi R = 2 \times 6.3 \times 10^{10} \text{ cm} \). The time it takes will be:

\[
T = \frac{2\pi R}{v} = \frac{2 \times 6.3 \times 10^{10} \text{ cm}}{102,000 \text{ cm/sec}} = 2.38 \times 10^6 \text{ sec} \approx 28 \text{ days}
\]

Problem: Derive a relation between a planet's period ("year") and its distance from the Sun.

Planets really go in ellipses about the Sun, not circles. However, we
will derive our relation by assuming that their orbits are circular. We will
do this (in good conscience) for three reasons: 1) To do it with ellipses
would take too long; 2) Circles will show the principles involved just as
well; 3) The planets' orbits are nearly circular. With this in mind:

Let \( M \) be the mass of the Sun, and let \( m \) be the mass of a planet. If the
planet is orbiting the Sun at a distance \( R \), then the formula we derived for
the Earth shows that its speed must be:

\[
V = \sqrt{\frac{GM}{R}}
\]

But \( V \) must also equal the length of the planet's orbit (\( 2\pi R \)) divided by
its period, which we will call \( T \). Therefore:

\[
V = \frac{2\pi R}{T}
\]

\[
\sqrt{\frac{GM}{R}} = \frac{2\pi R}{T}
\]

\[
\frac{GM}{R} = \frac{4\pi^2 R^2}{T^2}
\]

\( (**) \)

\[
R^3/T^2 = \frac{GM}{4\pi^2}
\]

For any planet in the Solar System, the quantity on the right is a
constant. So, for any planet:

\[
R^3/T^2 = \text{constant}
\]

This relation was discovered by Johannes Kepler. It is called Kepler's
Third Law of Planetary Motion. It does not just work for planets, nor just
for bodies whose orbits are circular. It is true for asteroids, comets, and
astronauts' gloves. And it works for non-circular orbits, if we let \( R \) equal
the body's mean radius (= average of its minimum and maximum distances from
the Sun).

From equation (**), we can calculate the mass of the Sun. Let the
\( R \) and \( T \) be those for Earth (respectively, \( 1.50 \times 10^{13} \text{ cm} \) and \( 31.5 \times 10^6 \text{ seconds} \)). The equation implies that:

\[
M = \frac{4\pi^2 R^3}{(GT^2)}
\]

\[
= \frac{4\pi^2 (1.50 \times 10^{13} \text{ cm})^3}{(1 \times (31.5 \times 10^6 \text{ sec})^2)}
\]

\[
= 2.01 \times 10^{30} \text{ gm}
\]

\[
= 2.01 \times 10^{27} \text{ metric tons}
\]

These last problems have been done by assuming that a large body (e.g.,
the Sun) stays fixed while a small body (e.g., the Earth) orbits around it.
This is not quite how things are. In reality, both the Earth and the Sun are
orbiting around a point that is between their centers, called their center
of mass. But the Sun is so much more massive than the Earth, that their
center of mass is practically at the center of the Sun. So it is alright for
us to pretend that it is.

In many cases, we can't do this. If, in a two-body system, one of the
masses is not overwhelmingly greater than the other, then we have to do the
problem the "right" way.
For two masses, \( m_1 \) and \( m_2 \), the center of mass is the point between them at which we should set the fulcrum, if we wanted to balance them on a lever. This is explained more clearly in the picture below.

If \( r_1 \) is the distance from \( m_1 \) to the center of mass, and \( r_2 \) the distance from \( m_2 \) to the center, then (by a problem worked in class), \( m_1 r_1 = m_2 r_2 \). Using this, we can do the following problem.

**Problem:** Two masses, \( m_1 \) and \( m_2 \), are orbiting in circles about their center of mass with constant frequency \( \omega \). The distance between them is \( R \). What is the sum of their masses? The situation is presented below.

The mass \( m_1 \) is going in a circle of radius \( r_1 \), with frequency \( \omega \). The force needed to keep it going in the circle is: \( m_1 \omega^2 r_1 \). The force which is keeping it there is the gravitational force from \( m_2 \), which is:

\[
G m_1 m_2 / R^2 = G m_1 m_2 / (r_1 + r_2)^2
\]

Therefore:

\[
m_1 \omega^2 r_1 = G m_1 m_2 / (r_1 + r_2)^2
\]

(1)

The next few equations must be followed carefully. Because \( m_1 \) and \( m_2 \) are orbiting around the center of mass:

\[
m_1 r_1 = m_2 r_2
\]

Adding \( m_2 r_1 \) to both sides of that equation gives:

\[
m_2 r_1 + m_1 r_1 = m_2 r_2 + m_2 r_1
\]

\[
(m_2 + m_1) r_1 = m_2 (r_2 + r_1) = m_2 R
\]

\[
r_1 = m_2 R / (m_1 + m_2)
\]

\[
1 / r_1 = (m_1 + m_2) / (m_2 R)
\]

Now we substitute this expression for \( 1 / r_1 \) back into (1), remembering that \( r_1 + r_2 = R \).
\[ \omega^2 = \left(\frac{Gm_2}{R^3}\right) \left(\frac{m_1 + m_2}{m_2R}\right) \]
\[ \omega^2 = \frac{G}{R^3} \frac{m_1 + m_2}{R^3} \]
\[ m_1 + m_2 = \frac{R^3}{\omega^2G} \]

REMARK

1. Two stars, of masses \( m \) and \( 5m \), are orbiting about their center of mass. The distance between them is one billion kilometers, and their period of revolution is one year. Find their masses, in kg.

2. Han Solo is once again flying the Millennium Falcon too close to the Death Star (a homogeneous sphere of mass \( M \)). When he is 1000 km from the Star's center, his engine's fail. He begins to gravitate toward it, with an initial acceleration of 10 m/sec\(^2\). What is \( M \), in kg?

3. The mass of the Moon is \( \frac{1}{81} \) times that of Earth. Its diameter is 1740 km. If a satellite orbits 10 km above the Moon's surface, what are its orbital period and orbital speed (in seconds; in m/sec)?

4. The distance from the Earth to the Moon (center to center) is 386,400 km. At the point \( P \), the gravitational pull of the Earth and Moon exactly balance. What is the distance \( L \)?

\[ \begin{array}{c}
E \quad 1 \quad \text{1000 km} \quad M \\
386 \quad P \\
\end{array} \]

5. How high off the Earth's surface must a satellite be to have an orbital period of 5 hours? Give your answer in kilometers.
In the last handout, a relation called Kepler's Third Law was mentioned. Now, there cannot be a Third Law without there also being a First and a Second Law. These other laws of Kepler's are given below:

I. Planets' orbits are ellipses, in which the Sun is at one focus.

II. The line joining a planet and the Sun sweeps out equal areas in equal times.

The meaning of the first law is clear. The meaning of the second is illustrated below.

The time between the points A and B on the planet's path is the same as the time between C and D. The second law says that the areas \( \overline{X} \) and \( \overline{Y} \) must also be equal. Intuitively, this means that the planet speeds up as it approaches the Sun, and slows down as it recedes from it. In terms of energy, this makes sense. When the planet is far from the Sun, it is "higher" (as seen from the Sun) than when it is near the Sun. Its potential energy is greater then. If its total energy is to stay constant, its kinetic energy (and hence, speed) must decrease as it goes out.

Kepler's first and third laws are consequences of the fact that gravity is an "inverse-square" force: it decreases with the inverse of the square of the distance between the interacting masses. The second law does not depend on that fact. The second law is true because gravity is a radial force—it acts along the straight line joining the two masses. So, it would be true even if the planets were connected to the Sun with springs.

To prove this, and with it the second law, we introduce a vector quantity called angular momentum, which we denote \( \mathbf{L} \) (ell, not one). If we set the Sun at the origin of our coordinate system, then the angular momentum of a planet (or anything) is defined as:

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p}
\]

where \( \mathbf{r} \) is the planet's (or anything's) position vector and \( \mathbf{p} \) is its momentum. This is diagrammed below.

If the planet is being acted upon by a force that is parallel to \( \mathbf{r} \), then \( \mathbf{L} \) does not change with time. To see this, we differentiate \( \mathbf{L} \), using the product rule for vectors:

\[
\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}
\]
Now, \( \frac{d\vec{r}}{dt} \) is the planet's velocity, \( \vec{v} \). This vector is parallel to \( \vec{p} \approx m\vec{v} \). Hence, their cross product is \( \vec{0} \). The equation reduces to:

\[
\frac{d\vec{L}}{dt} = \vec{r} \times (\vec{p} \times \frac{d\vec{p}}{dt})
\]

But \( \frac{d\vec{p}}{dt} \) is the force on the planet. Since it is radial—parallel to \( \vec{r} \)—the second cross product is also \( \vec{0} \). Therefore, \( \vec{L} \) is a constant.

The second law follows immediately from this fact. In the diagram below, the planet has swept a little area \( \Delta A \) in a short time \( \Delta t \). Here \( \Delta t \) is taken to be so small that the planet's velocity does not change significantly during it. For such a short time, we want to see what is: \( \frac{\Delta A}{\Delta t} \). Because, as we let \( \Delta t \) approach 0, this quantity will approach \( \frac{dA}{dt} \). The second law says that \( \frac{dA}{dt} \) is a constant.

\[
\vec{p} = mv
\]

![Diagram](image)

The area \( \Delta A \) is the area of triangle \( OXY \approx \frac{1}{2} \text{base} \times \text{height} \). The length of the base is \( \frac{1}{2} \Delta t \). The height is \( \sin \theta \) times the length of side \( XY \), which is \( \frac{1}{2} |\vec{r}| \Delta t \). Therefore:

\[
\Delta A = \frac{1}{2} |\vec{r}| \frac{1}{2} |\vec{r}| \sin \theta \Delta t
\]

\( \star \star \star \)

\[
\frac{\Delta A}{\Delta t} = \frac{1}{2} |\vec{r}| \frac{1}{2} |\vec{r}| \sin \theta
\]

From the same diagram, we see that the planet's angular momentum has a magnitude of:

\[
|\vec{L}| = |\vec{r}| |\vec{p}| \sin \theta
\]

\( \star \star \star \)

\[
|\vec{L}| = m |\vec{r}| |\vec{v}| \sin \theta
\]

where \( m \) is the planet's mass. By comparing equations (\( \star \)) and (\( \star \star \star \)), we find that:

\[
\frac{\Delta A}{\Delta t} = \frac{1}{2} \frac{|\vec{r}|}{m}
\]

\( \approx \) a constant, Q.E.D.

The second law is proved.

Problem: How much energy is needed (minimum) to blow up the world? This minimum energy is just enough to give all of the world's fragments sufficient energy to keep them from falling back together. In calculating it, we will ignore chemical and other forces that hold the world together; we will also pretend that the Earth is of uniform density.

As usual, we will begin by solving a general problem. If a mass \( m \) is sitting on the surface of a planet of radius \( R \) and mass \( M \), how much kinetic energy must the mass \( m \) be given in order to reach escape velocity?

In other words, what is the difference between such a mass's potential energy when it is on the planet and when it is an infinite distance away?
Giving the mass escape velocity means giving it just enough kinetic energy so that its speed will be zero "when it gets to infinity."

First, let us define its potential energy at infinity to be zero. Then the equation in Handout #16 (page 52) says that:

\[ U(\infty) - U(r_0) = \int_{r_0}^{\infty} F(r) \, dr \]

\[ = -U(r_0) \]

where \( F(r) \) is the gravitational attraction between \( m \) and the planet, and \( r_0 \) is the distance between \( m \) and the planet's center. If \( r_0 \) is greater than \( r_0 \), then:

\[ F(r) = -\frac{GMm}{r^2} \]

The force is negative because it is acting to decrease \( r \). Putting this into the integral yields:

\[ -U(r_0) = -\int_{r_0}^{\infty} \frac{GMm}{r^2} \, dr \]

\[ = GMm \int_{r_0}^{\infty} \frac{1}{r^2} \, dr \]

\[ = GMm/r_0 \]

\[ U(r_0) = -GMm/r_0 \]

The amount of energy needed is \( GMm/r_0 \). If \( m \) is on the planet, then it is \( GM/m \).

To calculate the energy needed to blow up the world, we break the Earth up into many infinitesimally thin shells. Using the above formula, we find the energy needed to blow away each shell. Summing (integrating) all these energies together will give us the total energy.

The diagram below shows a cutaway view of the Earth and a thin shell of radius \( r_0 \).

![Diagram of Earth and thin shell](image)

The thickness of the Earth is assumed to be uniform. We will call it \( \rho \) (rho) kg/m³. The thickness of the shell is \( dr \), and its surface area is \( 2\pi r \). Since this is a very thin shell, its volume will just be its surface area times its thickness: \( 2\pi r \, dr \). Its mass (density x volume) is \( 2\pi \rho r \, dr \).

The only mass that is pulling on it, giving it a negative potential energy, is the mass that is below it. It is as if the thin shell is resting on a small planet of radius \( r \) (it is, if the shells have been blown away). The volume of this "small planet" is \( 4/3\pi r^3/3 \). Hence its mass is \( 4\pi \rho r^3/3 \).

Letting the mass of the shell be \( m \), and the other mass be \( M \), our formula says that the energy needed to blow the shell away is:
\[
\frac{G M \pi r^3}{3} - \frac{16 \pi^2 \rho^2 G r^4}{3} dr
\]

We integrate this expression from 0 to R. The total energy is:

\[
E = \frac{(16/3) \pi^2 \rho^2 G r^4}{3} dr
\]

When we substitute \( M = \text{mass of the Earth} = 4\pi \rho^2 r^3 / 3 \) into this expression, we find, after some rather messy algebra:

\[
E = \frac{(3/5) GM^2}{R}
\]

\[
= 2.27 \times 10^{32} \text{ joules} = 2.27 \times 10^{39} \text{ ergs}
\]

That is how much energy would be released in an explosion of about \( 4.54 \times 10^{22} \text{ tons of TNT} \). Unless TNT production increases drastically, the Earth is safe, at least till Tuesday.

**Note:** 4.54 \( \times 10^{22} \text{ tons is} \approx 40 \text{ tons of TNT per cubic meter of earth. Since TNT weighs much less than 40 tons/m}^3 \text{, such an explosive charge would be larger than the Earth.**}
For this test you may use: handouts, class notes, old homeworks, old tests. Not every problem has the same point value. The problems' point values are given next to them. There are 30 points in all: 24 regular, plus 6 extra credit. Ration your time.

2pts. 1. Differentiate:
   a. \( \frac{x}{\sin^2 x + 1} \)  
   b. \( \frac{1}{\tan x + 1} \)

2pts. 2. Integrate:
   a. \( \int \frac{x^2}{(x^3 + 1)^3} \, dx \)
   b. \( \int (\sin x)(\cos x) \, dx \)

3pts 3. In the system below, the mass \( m \) slides down the ramp (of height \( h \), inclination \( \theta \)), across the horizontal stretch (of length \( L \)) and over the cliff of height \( h \). There is no friction, and the mass starts from rest. Find the time the mass spends in going from \( A \) to \( B \), and the distance \( x \) that it falls from the bottom of the cliff.

3pts 4. The masses \( m_1 \) and \( m_2 \) (\( m_1 < m_2 \)) are connected to the ends of a long massless cord. The cord has been hung through a frictionless loop \( E \). While \( m_2 \) hangs vertically, \( m_1 \) is swinging around it in a circle of constant radius, at a constant frequency \( \omega \). The distance between \( m_1 \) and \( E \) is \( L \), and \( m_2 \)'s part of the cord is an angle \( \theta \) out of the vertical. The mass \( m_2 \) is not moving up or down. Given \( m_1 \), \( m_2 \), and \( L \), what are \( \tan \theta \) and \( \omega \)?

3pts 5. The collision represented below is one-dimensional, and there is no friction. Find \( \frac{m_1}{m_2} \).

---

After

\[ \begin{array}{c}
\frac{3v}{m_1} \\
\frac{v}{m_2}
\end{array} \]
6. Far out in space, masses \( m_1 \) and \( m_2 \) are revolving about their common center of mass. They are connected to the ends of a massless spring of constant \( k \). The spring's unstretched length is \( x_0 \), but now its length is \( R > x_0 \). The gravitational attraction between \( m_1 \) and \( m_2 \) is negligible. Show that, if they are revolving with frequency \( \omega \), then:

\[
R = x_0 / \left( 1 - \frac{m_1 m_2 \omega^2}{k (m_1 + m_2)} \right)
\]

The distance between the masses is constant.

\[\text{Diagram showing masses and spring.}\]

7. A block is sliding along a horizontal surface under normal gravity. The coefficient of friction between the block and the surface is \( \mu \). The block's initial speed is \( v \). When it has gone a length \( L \), its speed has decreased to \( \alpha v \), where \( 0 < \alpha < 1 \). What is \( v \) in terms of \( \mu \), \( g \), \( \alpha \), and \( L \)?

\[\text{Diagram showing block and forces.}\]

8. Find the \( F(x, y) \) for each \( U(x, y) \):
   a. \( (x^2 - y^2) / (x^2 + y^2) \)
   b. \( \sin^2 x + \cos^2 y \)

9. Two asteroids, \( A \) and \( B \), are homogeneous spheres with respective radii \( R \) and \( 2R \). The acceleration of gravity on the surface of \( A \) is \( g \), and that for \( B \) is \( 2g \). If the density of \( A \) is \( \rho_1 \), and the density of \( B \) is \( \rho_2 \), what is \( \rho_1 / \rho_2 \)?

\[\text{Diagram showing two asteroids.}\]

Extra Credit

A. In the system to the right, the masses are as marked. The pulleys are massless and frictionless and the cords are massless. What is the acceleration of the mass inside the dotted circle (magnitude and direction)?

B. Four masses \( m \) are initially spinning with frequency \( \omega \) on the ends of massless rods of length \( L \). They are pulling up a cord and hook of negligible mass. The hook catches on a mass \( m_2 \), and the mass is lifted to some height. Then it falls back to earth. When it hits the ground, the other masses are
spinning at a frequency $\omega'$. What is the falling mass's speed when it hits the ground? (in terms of $\omega, \omega'$, etc.)
Relativity is actually a simple thing. It is best to introduce it by means of simple things.

Imagine an infinitely long, perfectly straight railroad track. At some point on the track, let us put a paint mark. That point shall be called "zero". One meter to the right of zero, let us put another mark. We shall call that point "one". One meter to the right of that, put another mark: two. And so on, out to infinity.

Now put a mark one meter to the left of zero. That point is minus one. One meter to the left of that, another: minus two. And so on, out to negative infinity.

Put a clock beside each mark on the track. Synchronize the clocks; then let them run.

This collection of paint marks and clocks is a means by which we can assign space and time coordinates to events that occur along the train track. For instance, suppose that a rabbit hops over the track, near the paint mark 15. When he hops over, the clock by paint mark 15 will read—let us say—one o'clock. Then the coordinates of this event (place and time) are:

\[(x, t) = (15, 1 \text{ o'clock})\]

Of course, the rabbit could have hopped exactly between two paint marks. For now, we will not worry about such situations. If we want to, we can put the paint marks closer together (marking off in half-meters or centimeters) and thus define an event's spatial position as accurately as we like. In any case, the position of an event (as measured by our system) will be the number on the paint mark that is closest to the event.

But we are not alone. There is a train on the tracks. It is as straight and as long as they. A man on the train has put paint marks and clocks, just like ours, along the whole length of the train. With these he can give any event that happens along the tracks space and time coordinates. When the rabbit jumps through the wheels of the train, it passes under his paint mark number 229; and clock number 229 reads one o'clock. The man gives this event coordinates:

\[(x', t') = (229, 1 \text{ o'clock})\]

The primes indicate that those are his numbers, and not ours. Actually, these clocks would not read 1 o'clock. We will use special clocks, which read pure numbers. The number that one of these clocks reads will be the number of seconds (possibly negative) from an arbitrarily chosen "zero time." The zero time is the temporal equivalent of the zero mark we put on the railroad track.

Is there any simple relation between the man's numbers \((x', t')\) and our numbers \((x, t)\) for a given event? In this fairy tale, yes. Let's say that the train's velocity—as measured by our marks and clocks—is \(v_c\) constant. Also, just as his zero mark passes over our zero mark, his clocks and our clocks all read zero (this would not have been hard to arrange). Then the simple relation between the numbers is:

\[
\begin{align*}
 x' &= x - vt \\
 t' &= t
\end{align*}
\]
That relation is easily explained. In this fairy tale, time is the same for everyone. Five hours passed on the train means five hours passed on the ground. Therefore, $t' = t$. If an event has coordinates $(x_1, t)$, then in the time $t$, the train has moved a distance $vt$ (if $t$ is positive; the situation when $t$ is negative is similar); the other man's zero mark has moved by $vt$. Then for him the paint mark near the event is not $x$, but $x- vt$.

Example: The train is going at 20 m/sec, in the positive direction. At time $t=5$ seconds, the rabbit hops across the track at mark number 120. But at $t=5$ seconds, the train's zero mark has moved from $x=0$ to $x=100$. When the rabbit hops across, he goes under mark number 20 (on the train).

And $20 < 120-100 \leq 120-(20)(5) = x-vt$. See the picture below.

![Diagram showing train and ground motion]

Relation (*) is called the Galilean transformation. It is the way that things "ought" to work, according to common sense. For everyday life, and at everyday velocities, it is the way they seem to work. It is not the way things work.

Here is why. The equation $t' = t$ says that events that are simultaneous for a man on the train are simultaneous for a man on the ground. Because, if two events are simultaneous, their time coordinates are equal, by definition. But time on the train is the same as time on the ground (we suppose). So simultaneity on the train implies simultaneity on the ground, and vice versa.

But this is false. This is shown in a parable, which Albert Einstein told.

Let us look at another train, which is moving at constant speed in a straight line. Three men are on the train, named A, B, and O'. There is a man standing on the ground, whose name is O. A and B are at opposite ends of the train, and are holding flashbulbs. O' is in the middle of the train. As O' passes O, each man sees two simultaneous flashes from the ends of the train. Each man wants to know: Who flashed first? See the diagram below.

![Diagram showing simultaneity on the ground and train]

Mr. O' thinks: "A and B are equidistant from me. I saw the flashes simultaneously. Hence, they must have flashed simultaneously."

Mr. O thinks: "Light travels at a finite speed. Therefore, A and B must have flashed their bulbs sometime before Mr. O' passed me, when the train was in a position like this: (TOP VIEW)"

"When the bulbs were flashed, A's flash had farther to travel than B's. It had to take longer to get to me. But the flashes reached me simultaneously. Therefore, A flashed before B."
Who is right? According to Einstein, both. In the reference frame of Mr. Q, A really flashed before B. In the other frame, the flashes were really simultaneous. They do not "appear" to be one or the other, but are. This is because time, like space, only has meaning with regard to a frame of reference. If I stand on my head, then the sky, in my frame of reference, is below my feet. So for O, A did flash before B.

A clever skeptic may object to Mr. O's argument like this: "It is true that A's flash had further to go than B's. But when A flashed, he was coming toward you, whereas B was going away from you. The light from A's flash—just like bullets from a gun—was given a boost toward you by A's speed. B's flash was given a boost away from you. A's flash came at you faster than B's, and was able to cover the greater distance in an equal time. The bulbs were flashed simultaneously."

The skeptic's argument is diagrammed below:

```
c= speed of light
```

The skeptic would be right, if he weren't wrong. In his argument, two flashes of light are going at different speeds. According to Einstein, this is impossible. The speed of light in a vacuum must be the same for all observers.

Why is this crazy fact true? It is a simple consequence of a sane fact. The same fact is: Physical laws are the same for all observers in all unaccelerated (important!) reference frames. This means that a man inside a sealed train that is moving in a straight line cannot perform any experiment which will tell him that he is moving. If he measures the voltage of a battery, or the time it takes for a rock to fall, he will get the same result as if he were standing still. If the windows are opened and he can see outside, there is no way for him to tell which is "really" moving: himself, or the landscape. If he sticks his hand out, the push of the air on it could be from him moving through the air, or a wind blowing past the train. There is no way for him to know.

Since experiments on the train must give the same results as those on the ground, if the man does any experiments to find the value of G, or any other fundamental physical constant (such as the charge on the electron, the half-lives of radioactive elements, Planck's constant), he must get the same number.

The speed of light—c—is such a constant. In the 19th Century, when James Clerk Maxwell derived the mathematical laws of electricity and magnetism, the speed of light appeared in them as a constant—"as if G is a constant in the law of gravitation. How this happened is too complicated for us to examine now. But the appearance of c in Maxwell's equations was the historical cause of relativity; if physical laws are the same for all, so are Maxwell's equations; and if the equations are the same for all, the speed of light is, too. Einstein's first relativity paper, published in 1905, was called "On the Electrodynamics of Moving Bodies."

Before going on, we will look at an experiment by which the man on the train—without turning on any flashlights or laser beams—could measure the speed of light. If two parallel wires are carrying electric current, they exert a force on each other. According to Maxwell, the force felt by each wire, per centimeter of wire length, is:

\[
\text{force/cm} = 2I_1 I_2 / (rc^2)
\]
where \( I_1 \) and \( I_2 \) are the currents in the wires, measured in electrostatic units (esu) per second, \( r \) is their separation in centimeters, and \( c \) is the speed of light, in centimeters per second. The force is in dynes.

The man could set up an experiment like the one at the left. He could measure with current meters; the force, with springs; the separation, with a ruler. He would measure the speed of light as being:

\[
c = \left( \frac{2I_1I_2}{rF} \right)^{\frac{1}{2}}
\]

\( F \) is the force/cm.

Obviously, he will get the same number on the train and on the ground.

But there are other, more obvious ways to measure the speed of light, and this is where things become interesting. The man could have measured \( c \) by measuring how long it took for a light beam to go a certain distance.

Let us return to our first situation, with the paint marks and clocks. A device which fires and absorbs laser beams (called a "laser-beam-firer-and-absorber") is standing beside the track. The laser gun is next to our zero mark, and the absorber is next to our \( x=L \) mark. Things have been set up as before: we have synchronized our clocks, and the man on the train has synchronized his. This time, however, we cannot say that our clocks are synchronized with his (simultaneity is not conserved). But we can set things up so that, when his zero mark passes our zero mark, \( b o t h \) of our "zero-mark clocks" will read zero.

With things so set up, we fire the laser when his zero mark passes ours. Both of our clocks (which are attached to computers) record the event. The event "firing of laser" has coordinates:

\[
(x,t) = (0,0)
\]

\[
(x',t') = (0,0)
\]

where the primes mean what they meant before. A little later, the beam is absorbed. Our clock at \( x=L \) records the event. The clock on the train which passes the absorber as the beam goes in (and which is not at \( x'=L' \)) also records it. The event "absorption of laser beam" has coordinates:

\[
(x,t) = (L,t)
\]

\[
(x',t') = (L',t')
\]

where we don't yet know \( L' \), \( t' \), or \( t' \).

The whole experiment is diagrammed below.
Since the speed of light is the same for everyone, we do know that \( \frac{L}{t_2} = c \), or that \( t_2 = \frac{L}{c} \). We also know that \( \frac{L'}{t_1} = c \). If we let \( \Delta x \) stand for \( x_2 - x_1 \), \( \Delta t \) for \( t_2 - t_1 \), \( \Delta t' \) for \( t_2' - t_1' \), and \( \Delta x' \) for \( x_2' - x_1' \), then the invariance of \( c \) implies that:

\[
(c \Delta t')^2 = (\Delta x')^2
\]

\[
(c \Delta t)^2 = (\Delta x)^2
\]

These equations can be re-written:

\[
c^2 (\Delta t')^2 - (\Delta x')^2 = 0
\]

\[
c^2 (\Delta t)^2 - (\Delta x)^2 = 0
\]

\[
c^2 (\Delta t')^2 - (\Delta x')^2 = c^2 (\Delta t)^2 - (\Delta x)^2
\]

If the temporal separation between two events (as seen in one frame of reference) is \( T \), and the spatial separation (as seen in the same frame) is \( X \), then the quantity:

\[
I = \left( c^2 T^2 - X^2 \right)^{\frac{1}{2}}
\]

is called the interval or proper time between the two events. In the above experiment, \( I = 0 \) for each observer. The interval between these two events must be zero for all observers (why?).

In order for \( I \) to equal zero on the ground and on the train, funny things have to happen to space and time. Our goal is to find out what these funny things are, in quantitative terms. We have one transformation—the Galilean—which we know to be incorrect. We need a new one, which will tell us how the space and time coordinates of an event really change as we go from one reference frame to another.

The interval will be a useful tool in our search. We need some terminology. If \( I^2 \) is 0, then the interval is called lightlike; there is just enough time between the two events for a light beam to have gone from one to the other (as in our laser experiment). If \( I^2 \) is positive, the interval is called timelike; there is more than enough time for a light beam to have passed between them. If \( I^2 \) is negative, the interval is spacelike; there is not enough time (and too much space) for the light beam to get from one to the other.

The interval is the same for all observers. This is obvious when the interval is lightlike. We will show that it is true for a timelike interval, too. Then we will use that fact to derive the new space-time transformation—called the Lorentz transformation. We will use the transformation to prove the invariance of spacelike intervals.

For simplicity, we will only consider reference frames that are moving in one dimension. That will be plenty.

To show the invariance of timelike intervals, first assume that two events have occurred, with coordinates \((0,0)\) and \((x_0,0)\), as seen in one frame. The square of the interval is \( c^2 t_0^2 - x_0^2 \). Since this is positive, there is more than enough time for a light beam to get from one event to the other; i.e., to leave the point \( x = 0 \) at \( t = 0 \), and arrive at \( x = x_0 \) when \( t = t_0 \). So to arrive on time, the light beam need not have gone straight from \( x = 0 \) to \( x = x_0 \); it could have gone in a bent path. On the next page, a way to do this is shown.
A mirror has been placed a height $h$ above $x=x_0/2$. The light beam leaves at $(0,0)$, bounces off the mirror, and arrives at $(x_0,t_0)$. The height $h$ has been adjusted so that the arrival of the beam and the event at $(x_0,t_0)$ are simultaneous (they will be simultaneous for anybody—why?).

Now we can evaluate $c^2t_0^2-x_0^2$. The distance the light beam travels is:

$$2((x_0/2)^2 + h^2)^{1/2}$$

It takes a time $t_0$ to do this, travelling with speed $c$. Therefore:

$$ct_0 = 2((x_0/2)^2 + h^2)^{1/2}$$

$$c^2t_0^2 - x_0^2 = 4h^2$$

The interval will be the same if the height of the mirror is the same for all observers. This question about the height is really two questions: 1.) Will the same mirror work in all frames? Could a mirror in a different place (as seen in our frame) work as ours did for someone in a different reference frame? 2.) Given that only a mirror at this height will work, will all observers get the same value for $h$?

The same mirror will do for everyone, because an event is an event is an event. Things happen, or they don't—for everybody—though the relations between those things may change. If someone could set a mirror halfway (spatially) between $(0,0)$ and $(x_0,t_0)$, but with a different height—as seen by us—and bounce the beam as we have, then so could we. Because the sequence of events: emission of beam, reflection, reception: has to occur in all frames. But since the speed of light is constant, our mirror is at the only height that will work.

Now for two. Heights (and widths) are the same for everyone. To see this, imagine one of our trains heading for a tunnel through which, when the train is moving very slowly, it just fits. Suppose that, for someone on the train, the tunnel shortens in height. Then the train must crash into the tunnel. But for someone standing by the tunnel, the train must appear to be shorter (for the same reason), and so there will be no crash. That's a contradiction: because the crash must occur or not occur for everyone. (One gets the same contradiction if the tunnel appears to be taller.)

For timelike intervals, the square of the interval will be $4h^2$, where $h$ is the height of the "special" mirror—which we have shown to be the same for all observers. Therefore, timelike intervals are invariant.
Relativity II

The invariance of timelike intervals is enough with which to derive the correct transformation for space and time coordinates. But before doing this, we shall make some small changes in our notation. These will make the algebra less messy.

From now on, we shall measure time in meters. One meter of time is defined as the time that it takes for a light beam to travel one meter. This is $3.336 \times 10^8$ seconds.

When time is measured like this, the interval between two events can be written:

$$(T^2 - X^2) \frac{1}{c^2}$$

where $T$ and $X$ have their obvious meanings.

If time is measured in meters, then velocity ($\approx \text{distance/time}$) is unitless. We will denote velocity by the Greek letter $\beta$. If $v$ is a velocity in conventional units, then the corresponding $\beta$ is:

$$\beta = \frac{v}{c}$$

In these units, the Galilean transformation is:

$$x' = x - \beta t$$
$$t' = t$$

We are looking for a new transformation. This transformation will have the form:

$$x' = Ax + Bt$$
$$t' = Cx + Dt$$

where $A$, $B$, $C$, and $D$ are constants. Such a transformation has the important property that, if $(x, t)$ transforms to $(x', t')$, then $(nx, nt)$—where $n$ is a number—transforms to $(nx', nt')$. Without this property, not all timelike intervals will be invariant.

We will find the constants $A$ through $D$ in two steps. Below is a diagram of an experiment. A train is moving with velocity $\beta < c$. At $t = 0$, $t' = 0$, its zero mark passes over the ground's zero mark. At this instant a light at $x = 0$ flashes.

The coordinates of this event are: $(x, t) = (x', t') = (0, 0)$. At time $t = t_0$ (as seen on the ground), another light flashes at $x = 0$. This event's coordinates are: $(x, t) = (0, t_0); (x', t') = (-\beta t_0, t_0)$; where we don't know (yet) what $t_0$ is. See the diagram below.
The interval between the two events is obviously timelike. Hence, it is the same for each reference frame. Hence:

\[ t_0' = t_0' - (\beta \cdot t_0')^2 \]
\[ = t_0 - \beta^2 t_0^2 \]

\((*)\)

\[ t_0' = t_0 / (1 - \beta^2)^{\frac{1}{2}} \]

If we let \(x_0'\) stand for the \(x'\)-coordinate of the second event, then we also have:

\[ x_0' = -\beta \cdot t_0 \]

\((**)*\)

\[ = -\beta t_0 / (1 - \beta^2)^{\frac{1}{2}} \]

Compare this to the transformation in the middle of page 71. When we substitute in the values \(x', t\)' etc., for the second event, we get:

\[ x_0' = Ax + Bt_0 = A(0) + Bt_0 \]
\[ = Bt_0 \]

\[ t_0' = Cx + Dt_0 = C(0) + Dt_0 \]
\[ = Dt_0 \]

The equations \((*)\) and \((**)\) tell us that:

\[ B = -\beta / (1 - \beta^2)^{\frac{1}{2}} \]

\[ D = 1 / (1 - \beta^2)^{\frac{1}{2}} \]

To find \(A\) and \(C\), we note that our transformation must satisfy:

\[ t'^2 - x'^2 = t^2 - x^2 \]

for a wide range of \((x, t)\) and \((x', t')\). This means that:

\[ (Cx + Dt)^2 - (Ax + Bt)^2 = t^2 - x^2 \]

\[ (D^2-B^2)t^2 + 2xt(CD-AB) - (A^2-C^2)x^2 = t^2 - x^2 \]

For this to be true for many, many \(x's\) and \(t's\), we must have that:

\[ CD-AB = 0 \]
\[ A^2 - C^2 = 1 \]

The top equation implies that \(C = A(B/D) = -\beta A\). Combined with the bottom equation, this gives us:

\[ A = 1 / (1 - \beta^2)^{\frac{1}{2}} \]
\[ C = -\beta / (1 - \beta^2)^{\frac{1}{2}} \]
The factor $\frac{1}{\sqrt{1 - \beta^2}}$ shows up in relativity so often that it has its own symbol. It is $\gamma$ (gamma; the Greek hard "g"). The Lorentz transformation, which we have just derived, is written:

\begin{align*}
    x' &= \gamma x - \beta \gamma t \\
    &= \gamma (x - \beta t) \\
    t' &= -\beta \gamma x + \gamma t \\
    &= \gamma (t - \beta x)
\end{align*}

The inverse Lorentz transformation is:

\begin{align*}
    x &= \gamma (x' + \beta t') \\
    t &= \gamma (t' + \beta x')
\end{align*}
Suppose we have two reference frames, \( S' \) and \( S \). The velocity of \( S' \) relative to \( S \) is \( \beta \). As seen in the \( S' \) frame, a particle is moving with a velocity \( \beta' \). What is its apparent velocity in \( S \)? The situation is diagrammed below:

\[
S' \quad \overset{\phi}{\longrightarrow} \quad S' (\text{in} \quad S') \quad \overset{\phi}{\longrightarrow} \quad S
\]

Let's suppose that the particle passes \( x=0 \) when \( t=0 \). Then its velocity in the \( S \) frame will be \( x/t \), for \( t \neq 0 \); where \( x \) is the particle's position, and \( t \) the time that it is at that position, as seen in the \( S \) frame. According to the Lorentz transformation, that quotient equals:

\[
\frac{\gamma(x'+\beta't')}{\gamma(t'+\beta'x')} = \frac{x'+\beta't'}{t'+\beta'x'}
\]

where \( x' \) and \( t' \) are the particle's coordinates in the \( S' \) frame.

Since the particle has velocity \( \beta' \) in the \( S' \) frame, and it starts from \( x'=0 \) when \( t'=0 \), we have a relation between the particle's \( x' \) and \( t' \):

\[
x' = \beta't'
\]

Substitute this into equation \((*)\) to get:

\[
\beta = \frac{\beta' + \beta t'}{t' + \beta \beta' \beta'}
\]

\[
= \frac{\beta' + \beta}{1 + \beta \beta'}
\]

This differs significantly from the "Newtonian" \( \beta + \beta' \) only when \( \beta \) and \( \beta' \) become of order 1. If \( \beta \) and \( \beta' \) are (in conventional units) each 10,000,000 mph, then the deviation from the Newtonian value is less than half a mile per hour.

The formula can be used to prove the impossibility of travelling faster than light.

Consider the experiment below. Mr. A has a gun that shoots bullets which have a \( \beta \) (relative to the gun) of 3. Mr. B is going by A with a relative velocity of \( \frac{1}{3} \). What is the bullet's \( \beta \) in his frame?

\[\text{Mr. A's } \beta, \text{ relative to } B, \text{ is } -\frac{1}{3}. \text{ According to the formula, then, the bullet's velocity, as seen by } B, \text{ will be: } (3 - \frac{1}{3})/(1 - 3 \times \frac{1}{3}) = -5. \text{ The bullet goes backwards, from the target into the gun.}

In A's frame, the event "bullet in gun" is the cause of the event "bullet in target." In B's frame, it is the opposite. And here is where we get
our contradiction. The bullet has to go into the gun in B's frame, because it leaves the gun in A's frame. But there is nothing to stop B from sticking out a bucket as it passes by (he needn't even see it to do it) and catching it. It doesn't go into the gun: contradiction. Therefore, hyperlight travel is impossible—at least for anything that interacts with matter.

The coordinates of events can be recorded on a graph, like the one below. The numbers on the horizontal axis give the positions of events; those on the vertical axis give the times. In this reference frame, events A and B are simultaneous, and A and C occur at the same place.

An observer with some velocity relative to this frame can also plot these events on a space-time graph. His numbers, and the relations between the events, will be different. To understand approximately how things will look for him, we will draw a Minkowski diagram.

In the first (stationary) frame, the x-axis can be identified with all those events whose time-coordinate is 0. Likewise, the t-axis is those events whose space-coordinate is 0. For the moving observer, the analogous thing will be true of his x'- and t'-axes.

The Lorentz transformation (page 73) says that \( \frac{x'}{c} = 0 \) when \( t = \beta x \). And \( x' = 0 \) when \( x = \beta t \). Therefore, on the stationary observer's graph, we identify the moving observer's x'- and t'-axes with the lines \( x = \beta t \) and \( t = \beta x \).

For \( \beta = \frac{1}{2} \), the Minkowski diagram looks like this. Events simultaneous for the moving observer are on the same line that is parallel to the x'-axis; in his frame, B occurs before A. Events which occur in the same place (for him) are on the same line parallel to the t'-axis.

The curved line in the diagram represents the history of a moving particle. It is called the particle's world line, and it consists of all the events along the particle's path.

Using Minkowski diagrams, we can show that another popular device among science (sic) fiction writers—namely, hyperspace travel—is impossible.

For those who do not read science fiction, the idea (is) behind hyperspace travel is this: Granted that one cannot go faster than light, one travels between stars by simply "jumping" from place to place. To go from Earth to Sirius, one pushes a button on a machine—a deus ex machina—disappears, and an instant later reappears near Sirius.

If such were possible, then you could do the following. First, jump a great distance; 100 light-years, for example. The two events, "disappear near Earth" and "reappear wherever" are simultaneous in your reference frame. Then accelerate to some not-too-large speed (the further you jump, the smaller it has to be). Now those events are no longer simultaneous for you. The event "disappear..." is in the future, in your frame. Jump back to your starting place. As your ship is about to enter hyperspace, destroy it with your photon torpedoes. Then you never enter hyperspace; but then you never come back to destroy your ship; but then you do enter hyperspace; and you do destroy your ship; and you don't enter hyperspace... See the Minkowski diagram below.
STAR WARS QUIZ

1. Who said each of the following, and under what circumstances?
   a. "Thank the maker!"
   b. "I sense something; a presence I have not felt since . . . ."
   c. "Their fire has gone out of the Universe."
   d. "Only a master of evil, Darth."

2. What weapon was used to destroy the Death Star? Why?

3. Han Solo owed money to _______. Why did Solo owe him money? What had he(Solo) done?

4. What two facts indicated that it was Imperial storm troopers, and not Sand People, who attacked the jawas' transport?

5. What was Toffkin's official title?

6. Where were the outside scenes of the rebel base filmed?

7. The rebel fighters were too small for the Death Star's _______ _______ to be effective against them(specific weapon).

8. (Extra credit) In how many years did the Millennium Falcon make the Kessel run?

9. (Extra credit) If the Death Star builders were so smart, why wasn't there a safety grating on the exhaust port?
1. How useful were the:
   a. handouts
   b. lectures
   c. homework problems

   in helping you understand the material?

2. How was the pace? Too fast? Too slow? Okay?

3. How many hours/night (average) did you spend on this class? Did that seem like too much, too little, or what?

4. Were the tests fair? How well did they test material from the class?

5. What grade did you receive? What grade do you think you deserved?

6. What criticisms do you have of the class? Suggestions for improvement?

7. What was good about the class?