The Natural Numbers Add Up to $-1/12$ or Possibly $-1/16$

Michael Wilson
Department of Mathematics
University of Vermont

By now many of us have seen the Youtube video\(^1\) in which a physicist claims to show that $1 + 2 + 3 + 4 + \cdots = -1/12$. Briefly, his argument is this. Set:

\[
\sigma_1 = \sum_{1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 - 1 + \cdots \\
\sigma_2 = \sum_{1}^{\infty} (-1)^{n+1} n = 1 - 2 + 3 - 4 + 5 - \cdots \\
\sigma_3 = \sum_{1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots .
\]

The partial sum $\sum_{1}^{N} (-1)^{n+1}$ equals 1 if $N$ is odd and 0 if $N$ is even. To get $\sigma_1$ (the sum for $N = \infty$) one takes the average; thus, $\sigma_1 = 1/2$. Now add $\sigma_2$ to itself, after shifting all the terms over by one, to get

\[
2\sigma_2 = 1 + (-2 + 1) + (3 - 2) + (-4 + 3) + (5 - 4) + \cdots \\
= 1 - 1 + 1 - 1 + 1 \cdots \\
= 1/2,
\]

implying $\sigma_2 = 1/4$. (We’ll see a second way to “derive” this soon.) To finish, take the difference $\sigma_3 - \sigma_2$,

\[
\sigma_3 - \sigma_2 = (1 + 2 + 3 + 4 + 5 + \cdots) - (1 - 2 + 3 - 4 + 5 - \cdots) \\
= 4 + 8 + 12 + 16 + \cdots
\]

(because the odd terms cancel and the evens get doubled), which is $4(1 + 2 + 3 + 4 + 5 + \cdots) = 4\sigma_3$, yielding

\[
\sigma_3 - \sigma_2 = 4\sigma_3 \\
3\sigma_3 = -\sigma_2 \\
\sigma_3 = -(1/3)\sigma_2 = -1/12.
\]

What is really going on here? How is it even possible to add up all of the positive integers to get a negative quantity? The secret lies in an abuse of something called *Abel summation*.

Consider the first sum $\sigma_1$. Rather than add it up as is, we can “sneak up” on it by multiplying each term by a power of $x$, where $x$ will be a number between 0 and 1. We’ll call the resulting sum (which depends on $x$) $\sigma_1(x)$:

\[
\sigma_1(x) = x - x^2 + x^3 - x^4 + x^5 + \cdots \\
= \sum_{1}^{\infty} (-1)^{n+1} x^n.
\]

\(^1\) http://www.youtube.com/watch?v=w-I6XTVZXww
As long as $0 < x < 1$, the sum $\sigma_1(x)$ (a variation on something called the geometric series) adds up to a number, and that number happens to be

$$\frac{x}{1 + x}.$$

If we let $x \to 1$ from below, then $\frac{x}{1 + x}$ approaches $1/2$; while, in a sense, $\sigma_1(x) \to \sigma_1$ (because all the $x^n$ factors go to 1). Thus, in a sense,

$$\sigma_1 = \lim_{x \to 1^-} \sigma_1(x) = \lim_{x \to 1^-} \left( \frac{x}{1 + x} \right) = 1/2. \quad (1)$$

In the preceding extended equation, only the second and third equalities are literally true. In mathematical parlance, we say that the infinite series of $\sigma_1$ is Abel summable to $1/2$. In general we say that an infinite series $\sum_{1}^{\infty} a_n$ is Abel summable to $L$ if: (a) for every $0 < x < 1$, the series $\sum_{1}^{\infty} a_n x^n$ really sums (with no funny business) to a number; and (b) as $x \to 1^-$, the series $\sum_{1}^{\infty} a_n x^n$ approaches $L$. Abel summation is often better behaved (i.e., can be applied to more series) than ordinary summation. This is because, if $0 < x < 1$, then $x^n \to 0$ very fast as $n \to \infty$, so that most of the terms in the series $\sum_{1}^{\infty} a_n x^n$ get heavily damped down. As $x \to 1^-$ the damping factors $x^n$ go to 1, but at different rates, so that each individual term $a_n$ (so to speak) gets brought into the sum gradually instead of all at once: the summation is “smoothed out”.

As long as $0 < x < 1$, we can take the derivative of $\sigma_1(x)$ in the obvious (term-by-term) sense, to get:

$$\sigma_1'(x) = \sum_{1}^{\infty} (-1)^{n+1} n x^{n-1},$$

which also equals

$$\left( \frac{x}{1 + x} \right)' = \frac{1}{(1 + x)^2}.$$

We now multiply both of these by $x$ to get

$$x \sigma_1'(x) = \sum_{1}^{\infty} (-1)^{n+1} n x^n = \frac{x}{(1 + x)^2}.$$

As $x \to 1$ from below, the sum in the middle looks more and more like the expression we called $\sigma_2$ (which we can see by setting $x = 1$). Thus, in a sense,

$$\sigma_2 = \lim_{x \to 1^-} x \sigma_1'(x) = \lim_{x \to 1^-} \frac{x}{(1 + x)^2} = 1/4. \quad (2)$$

As with (1), only the second and third equalities in (2) are literally true. We say that the infinite series of $\sigma_2$ is Abel summable to $1/4$. We set

$$\sigma_2(x) = \sum_{1}^{\infty} (-1)^{n+1} n x^n$$

for $0 < x < 1$. Equation (2) says that $\lim_{x \to 1^-} \sigma_2(x) = 1/4$.

What about $\sigma_3$? Is it Abel summable to $-1/12$? NO. If we take

$$\sum_{1}^{\infty} n x^n$$
for $0 < x < 1$ and let $x$ approach 1, the series approaches infinity. Getting $-1/12$ requires more trickery. We put

$$s_3(x) = \sum_{1}^{\infty} nx^n$$

for $0 < x < 1$, and look at $s_3(x) - \sigma_2(x)$. It is:

$$s_3(x) - \sigma_2(x) = \sum_{1}^{\infty} (n - (-1)^{n+1})x^n$$

$$= \sum_{1}^{\infty} n(1 - (-1)^{n+1})x^n.$$

Now, $1 - (-1)^{n+1}$ equals $1 - 0 = 1$ if $n$ is odd and equals $1 - 2 = -1$ if $n$ is even. Therefore

$$s_3(x) - \sigma_2(x) = \sum_{n: n \text{ even}} 2nx^n.$$

But if $n$ is even then $n = 2k$ for some $k$, so we can rewrite the last series as

$$\sum_{1}^{\infty} 2(2k)x^{2k},$$

or (changing the index back to $n$),

$$\sum_{1}^{\infty} 4nx^{2n}.$$

Putting it all together,

$$s_3(x) - \sigma_2(x) = \sum_{1}^{\infty} 4nx^{2n},$$

or

$$\sum_{1}^{\infty} 4nx^{2n} - s_3(x) = -\sigma_2(x).$$

Rewrite the left-hand side of (4) in terms of infinite series (see (3)) to get

$$\sum_{1}^{\infty} 4nx^{2n} - \sum_{1}^{\infty} nx^n = \sum_{1}^{\infty} n(4x^n - 1)x^n.$$

When we plug this back into (4) we have

$$\sum_{1}^{\infty} n(4x^n - 1)x^n = -\sigma_2(x),$$

which (after dividing by 3) gives us

$$(1/3)\sum_{1}^{\infty} n(4x^n - 1)x^n = -(1/3)\sigma_2(x).$$
We will refer to the left-hand side of (5) as $\sigma_3(x)$. As $x$ approaches 1 from below, each expression $(4x^n - 1)x^n$ approaches 3, but we’re also dividing the whole series by 3; so that, in a sense, the whole infinite series of $\sigma_3(x)$ looks more and more like

$$
(1/3) \sum_{n=1}^{\infty} 3n = \sigma_3,
$$

and we have

$$
\sigma_3 = \lim_{x \to 1^-} \sigma_3(x) = \lim_{x \to 1^-} -(1/3)\sigma_2(x) = -1/12,
$$

where, once again, only the second and third equalities are literally true.

Such is the mathematical basis of “$1+2+3+4+5+\cdots = -1/12$”. Now we see how the peculiar method of summation in $\sigma_3(x)$ (“Abel with gimmicks”) yields a negative result, even though all the natural numbers are positive. Observe that, for any $0 < x < 1$, the multiplying factors $(4x^n - 1)x^n$ will be negative for all but finitely many $n$’s (because $4x^n \to 0$ as $n \to \infty$). Thus, for every $0 < x < 1$, the series $\sigma_3(x)$ will have finitely many positive, but infinitely many negative terms.

Here’s one last bit of fun. If $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ equals 1/2, then so should $1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - \cdots$, since all we have done is put in a lot of zeroes. Let’s add this to $\sigma_3$ as follows:

$$
\sigma_3 + 1/2 = (1 + 2 + 3 + 4 + \cdots) + (1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + 0 - \cdots)
= 1 + (2 + 1) + (3 + 0) + (4 - 1) + (5 + 1) + (6 + 0) + (7 - 1) +
(8 + 1) + (9 + 0) + (10 - 1) +
= 1 + [(2 + 1) + (3 + 0) + (4 - 1)] + [(5 + 1) + (6 + 0) + (7 - 1)] +
[(8 + 1) + (9 + 0) + (10 - 1)] + \cdots
= 1 + 9 + 18 + 27 + \cdots
= 1 + 9(1 + 2 + 3 + 4 + \cdots)
= 1 + 9\sigma_3,
$$

implying

$$
\sigma_3 + 1/2 = 1 + 9\sigma_3
$$

$$
8\sigma_3 = -1/2
$$

$$
\sigma_3 = -1/16.
$$

(6)

The equation “$\sigma_3 = -1/12$” is used in string theory\(^2\). Its occurrence in physics raises two questions.

One: Among the infinity of ways to regularize divergent series, why does nature choose a form based on Abel summation?

Two: As equation (6) shows, adding up a divergent series—with its canceling of infinities and negative infinities—is a delicate, unstable operation. If “$\sigma_3 = -1/12$” makes sense physically, what does that say about the (extreme!) fine tuning of the Universe?

---